CSPs over structures with slow unlabelled growth

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AAA 102 Szeged, 25th June 2022

This project has received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No 771005)

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Alternative formulation

- INPUT: A primitive positive sentence φ over $\mathfrak{B}$ ($\varphi \equiv \exists\exists\ldots\exists(\bigwedge(\text{atomic})))$
- QUESTION: Is φ true in $\mathfrak{B}$?

Constraint Satisfaction Problems

Finite domain dichotomy

Theorem (Bulatov; Zhuk '17)

If $\mathfrak{A}$ is finite then $\mathrm{CSP}(\mathfrak{A})$ *is in* **P** *or it is* **NP***-complete.*

Conjecture (Bodirsky, Pinsker '11)

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Remark: all these structures are ω-categorical.

Definition

$\mathfrak{A}$ is ω-categorical if $\mathrm{Aut}(\mathfrak{A})$ has finitely many n-orbits for all $n \in \omega$.

Constraint Satisfaction Problems

Infinite domain dichotomy

CSP dichotomy is solved for

- reducts of $(\mathbb{N}, =)$ (Bodirsky, Kára '08)
- redcuts of $(\mathbb{Q}, <)$ (Bodirsky, Kára '09)
- reducts of the homogeneous binary branching C-structure (Bodirsky, Jonsson, Pham '16)
- reducts of the random graph (Bodirsky, Martin, Pinsker, Pongrácz '19)
- reducts of the random poset (Kompatscher, Pham '18)
- reducts of unary ω-categorical structures (Bodirsky, Mottet '18)
- MMSNPs (Bodirsky, Madelaine, Mottet '18)
- reducts the random tournament (Mottet, Pinsker '21)
- first-order expansions of the homogeneous RCC5 structure (Bodirsky, B. '21)
- hereditarily cellular structures (B. '22)
- first-order expansions powers of $(\mathbb{Q}, <)$ (Bodirsky, Jonsson, Martin, Mottet, Semanišinová)
- reducts of random uniform hypergraphs (Mottet, Nagy, Pinsker)

Orbit growth function

Definition

$\mathfrak{A}$: structure.

- $o_n(\mathfrak{A}) := \#\{n\text{-orbits of } \mathfrak{A}\} = \#\{\text{orbits of } \mathrm{Aut}(\mathfrak{A}) \curvearrowright A^n\}$

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General question:
Given some upper bound $f : \omega \to \mathbb{R}$, determine all structures with $\ell_n(\mathfrak{A}) < f(n)$ or $u_n(\mathfrak{A}) < f(n)$.

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Stability

Definition

A formula $\phi(\bar{x}, \bar{y})$ has the order property (in $\mathfrak{A}$) if for all $\exists(\bar{a}_j, \bar{b}_j : j \in \omega)$ such that $\mathfrak{A} \models \phi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j$. ("$\phi$ defines a half-graph")

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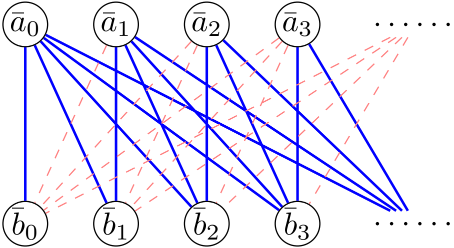

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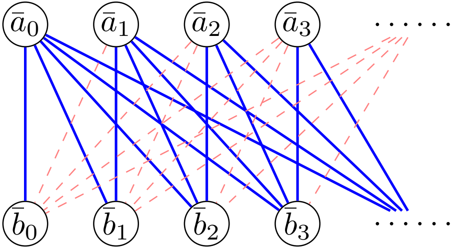

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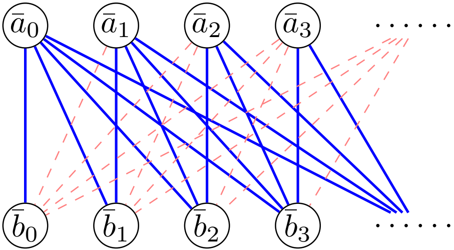

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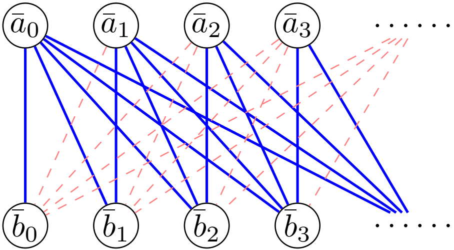

- Stable structures: pure set, unary structures, vector spaces,...
- Unstable structures: $(\mathbb{Q}; <)$, random graph, infinite Boolean algebras,...

Stable case

Orbit growth dichotomy

Theorem (Braunfeld '22)

$\mathfrak{A}$: *ω-categorical stable structure. Then one of the following holds.*

1. $\forall c > 1\,(u_n(\mathfrak{A}) < c^n)$.
2. $\forall c\,(u_n(\mathfrak{A}) > c^n)$.

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1. $\forall c > 1\,(u_n(\mathfrak{A}) < c^n)$.
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Moreover the 1 holds if and only if $\mathfrak{A}$ is hereditarily cellular (see next slide).

Hereditarily cellular structures

Recursive description

Definition/Theorem (B. '21+Lachlan '92)

$\mathfrak{A}$ is hereditarily cellular iff it can be constructed from finite structures by taking

1. finite disjoint unions,
2. infinite copies,
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Hereditarily cellular structures

Recursive description

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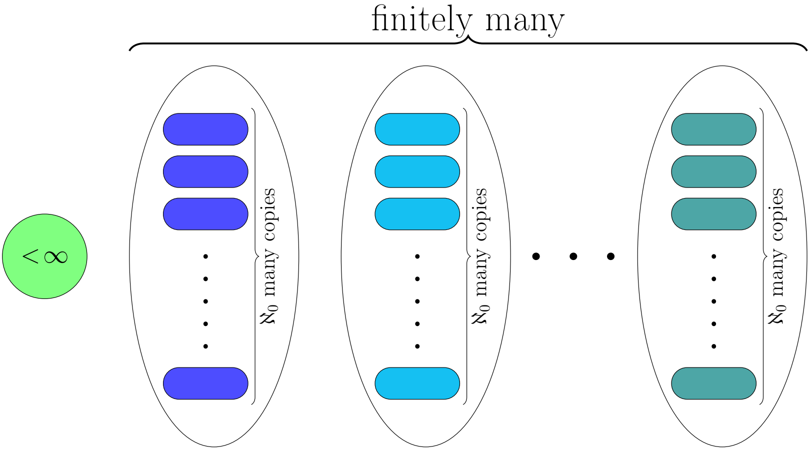
- Infinite copies: we add an equivalence relation E whose equivalence classes are the copies

Hereditarily cellular structures

Illustration



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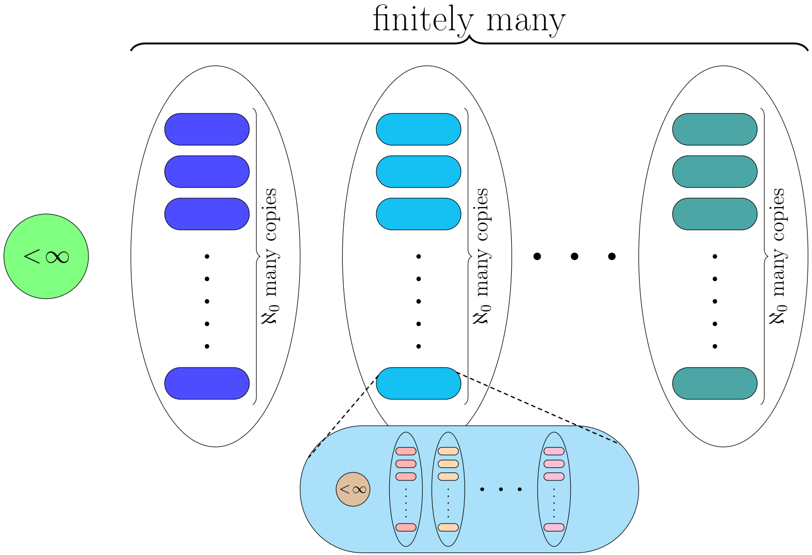

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They all fall into the infinite-domain dichotomy conjecture.

Hereditarily cellular structures

Getting back to CSPs

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Theorem (B. '21)

For every hereditarily cellular structure $\mathfrak{A}$ *(with a finite relational signature)* $\mathrm{CSP}(\mathfrak{A})$ *is in* **P** *or* **NP***-complete.*

Idea of the proof:

- By general results (Barto, Pinsker) we can assume that $\mathfrak{A}$ has pseudo-Siggers polymorphism:
  $(\alpha \circ f)(x, y, x, z, y, z) = (\beta \circ f)(y, x, z, x, z, y) : \alpha, \beta \in \mathrm{End}(\mathfrak{A})$

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Remark: this always works for hereditarily cellular structures (heaven).

Structures with subexponential unlabelled growth

Removing stability...

Theorem (B. '22)

$u_n(\mathfrak{A}) < c^n$ for all $c > 1$ iff it can be constructed from finite structures and $(\mathbb{Q}, <)$ by taking

1. *finite disjoint unions,*
2. *infinite copies,*
3. *first-order reducts.*

Structures with at most $(2-\varepsilon)^n$ unlabelled growth

Theorem (B. '22)

If $u_n(\mathfrak{A}) < c^n$ for some $c < 2$ iff it can be constructed from finite structures and "some" finite covers of $(\mathbb{Q}, <)$ by taking

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Corollary (B. '22)

These structures are all interdefinable with a finitely bounded homogeneous structure.

We are still in scope of the CSP dichotomy conjecture.

Structures with at most $(2-\varepsilon)^n$ unlabelled growth

Growth rates

Theorem

Assume $u_n(\mathfrak{A}) < c^n$ for some $c < 2$. Then one of the following holds.

1. *$u_n(\mathfrak{A})$ is slower than exponential.*
2. *$c_1\gamma_d^n < u_n(\mathfrak{A}) < c_2(\gamma_d + o(1))^n$ for some $2 \leq d \in \omega$ where γ_d is the largest real root of the polynomial $x^d - x^{d-1} - \cdots - 1 = 0$.*

$\gamma_2 \approx 1.618$
$\gamma_3 \approx 1.839$
$\gamma_d \nearrow 2$

Structures with at most $(2-\varepsilon)^n$ unlabelled growth

CSPs again

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general reduction to finite domain CSPs never works in the unstable case (hell)!

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Theorem (Bodirsky, Kára '09; Mottet, Pinsker '21)

*Let $\mathfrak{A}$ be a first-order expansion of $(\mathbb{Q},<)$. Then if $\mathrm{CSP}(\mathfrak{A})$ is not **NP**-complete then $\mathrm{Pol}(\mathfrak{A})$ contains pp, ll, dual-pp, dual-ll or a constant operation.*

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- ω-categorical unary expansions of $(\mathbb{Q}, <)$?

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- homogeneous C- and D-structures
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What are the possible growth rates of structures with $u_n < c^n$?

Further questions

What can we say about structures with at most c^n unlabelled growth? More examples:

- finite covers of $(\mathbb{Q}, <)$
- $S(2)$, the *local order*, $u_n \sim 2^{n-1}/n$
- homogeneous C- and D-structures
- we can take finite covers and expansions with *random partitions*

Conjecture (Braunfeld)

The class $\{\mathfrak{A} : u_n(\mathfrak{A}) < c^n \text{ for some } c\}$ is closed under taking ω-categorical unary expansions.

What are the possible growth rates of structures with $u_n < c^n$?

Conjecture

If $u_n(\mathfrak{A}) < c^n$ then $\mathfrak{A}$ is finitely homogenizable.