Harmonikus sokaságok és a görbék körüli csövek térfogata

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Harmonic Manifolds

Definition

A Riemannian manifold is a (locally) harmonic space if the following equivalent definitions are fulfilled.

- Each point $p \in M$ has a neighborhood on which the equation $\Delta u = 0$ has a non-constant solution of the form $u(q) = f(d(p,q))$, where $f\colon (0,a) \to \mathbb{R}$ is real analytic on $(0,a)$. [Ruse 1930].
- At each point $p \in M$, the volume density function $\theta_p = \sqrt{\det(g_{ij})}$ written in normal coordinates centered at p is radial, i.e., $\theta_p(\mathbf{v}) = \bar{\theta}(\|\mathbf{v}\|)$ for some function $\bar{\theta}$.
- Every sufficiently small geodesic sphere has constant mean curvature.
- (if $\dim M > 2$) Every sufficiently small geodesic sphere has constant scalar curvature.
- If f is harmonic on a neighborhood of a sufficiently small geodesic ball $B(p,r)$, then

$$f(p) = \frac{1}{\mathrm{Vol}(S(p,r))} \int_{S(p,r)} f \mathrm{d}\sigma.$$

Examples of Harmonic Spaces

Definition

A metric space (X, d) is called 2 point homogeneous if for any $p, q, p', q' \in X$ such that $d(p, q) = d(p', q')$, there is an isometry $\Phi : X \to X$ such that $\Phi(p) = p'$ and $\Phi(q) = q'$.

Proposition

A connected Riemannian manifold is 2 point homogeneous if and only if it is complete and its isometry group acts transitively on the bundle of unit tangent vectors. In particular, spaces locally isometric to a 2 *point homogeneous space are harmonic.*

Theorem (J. Tits, H.C. Wang, Z.I. Szabó)

Connected 2 *point homogeneous Riemannian manifolds are*

- *the Euclidean spaces* $\mathbf{E}^n$,
- *the simply connected rank* 1 *symmetric spaces*
  $$\mathbf{S}^n,\ \mathbb{C}\mathbf{P}^n,\ \mathbb{H}\mathbf{P}^n,\ \mathbb{O}\mathbf{P}^2,\quad \mathbf{H}^n,\ \mathbb{C}\mathbf{H}^n,\ \mathbb{H}\mathbf{H}^n,\ \mathbb{O}\mathbf{H}^2,$$
- *the real projective spaces* $\mathbb{R}\mathbf{P}^n$.

Examples of Harmonic Spaces

Lichnerowicz Conjecture

Every harmonic space is locally isometric to a 2 point homogeneous space.

Theorem (Z.I. Szabó)

If a simply connected and connected harmonic space is compact, then it is a rank 1 symmetric space.

- In the non-compact case, the Lichnerowicz conjecture is false.

Theorem (E. Damek, F. Ricci)

There are many non-symmetric harmonic spaces among solvable extensions of Heisenberg type 2-step nilpotent Lie groups equipped with left invariant Riemannian metrics.

Theorem (J. Heber)

Every homogeneous harmonic space is locally isometric to a 2 point homogeneous space or a Damek–Ricci space.

Tubes about a Curve

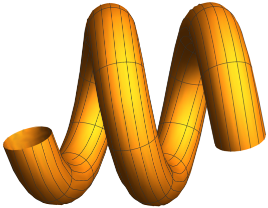
- (M, g) – connected Riemannian manifold;
- $\exp\colon \mathring{T}M \to M$ – its exponential map;
- $\gamma\colon [a, b] \to M$ – an injective regular curve;
- For $r > 0$, set

$$T(\gamma, r) = \bigcup_{t\in[a,b]} \{\mathbf{v} \in T_{\gamma(t)}M, \mathbf{v} \perp \gamma'(t), \text{ and } \|\mathbf{v}\| \leq r\}.$$

Definition

Assume that r is small enough to guarantee that the exponential map is defined and injective on $T(\gamma, r)$. Then we define the tube of radius r about γ by

$$\mathcal{T}(\gamma, r) = \exp(T(\gamma, r)).$$



The Tube Property

Theorem (Hotelling)

In the Euclidean and spherical spaces, the volume of $\mathcal{T}(\gamma, r)$ depends only on the length of γ and the radius r.

Theorem (H. Weyl)

The volume of a tube of radius r about a submanifold of $\mathbf{E}^n$ or $\mathbf{S}^n$ depends only on intrinsic invariants of the submanifold and on r.

Definition

- We say that a Riemannian manifold has the tube property if there is a function $V : [0, \infty) \to \mathbb{R}$ such that

$$\mathrm{Vol}(\mathcal{T}(\gamma, r)) = V(r) l_\gamma \tag{1}$$

  for any smooth injective regular curve γ of length l_γ and any sufficiently small r.
- the space has the tube property for geodesics if (1) holds for any injective geodesic arc and any small radius. r.

Theorem (A. Gray, L. Vanhecke, 1982)

Every rank 1 symmetric space has the tube property. (In these spaces, the volume of tubes were computed explicitly.)

Main Theorem

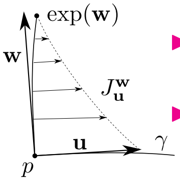
For a connected Riemannian manifold, the following properties are equivalent

- *the manifold is harmonic;*
- *the manifold has the tube property;*
- *the manifold has the tube property for geodesic curves.*

In a harmonic space, the volume of a tube of radius r about a curve of length l_γ is

$$\omega_{n-1} r^{n-1} \bar{\theta}(r) l_\gamma.$$

Formula for the Volume of a Tube



- For $\mathbf{0} \neq \mathbf{u} \in T_pM$ set $B_r^{n-1}(\mathbf{u}) = \{\mathbf{w} \in T_pM : \mathbf{w} \perp \mathbf{u},\ \|\mathbf{w}\| \leq r\}$.
- For $\mathbf{0} \neq \mathbf{u} \in T_pM$ and $\mathbf{0} \neq \mathbf{w} \in B_r^{n-1}(\mathbf{u})$, let $J_{\mathbf{u}}^{\mathbf{w}}$ be the Jacobi field along the geodesic $t \mapsto \exp(t\mathbf{w})$ defined by

$$J_{\mathbf{u}}^{\mathbf{w}}(0) = \mathbf{u} \quad \text{and} \quad J_{\mathbf{u}}^{\mathbf{w}}(1) = \mathbf{0}.$$

Theorem

The volume of the tube of radius r about the unit speed curve $\gamma \colon [a,b] \to M$ equals

$$-\int_a^b \int_{B_r^{n-1}(\gamma'(t))} \big(\langle {J_{\gamma'(t)}^{\mathbf{w}}}'(0), \gamma'(t)\rangle + \langle \gamma''(t), \mathbf{w}\rangle\big)\theta(\mathbf{w})\mathrm{d}\mathbf{w}\mathrm{d}t.$$

Theorem

The volume of the tube of radius r about the unit speed curve $\gamma\colon [a,b] \to M$ equals

$$-\int_a^b \int_{B_r^{n-1}(\gamma'(t))} \big(\langle {J^{\mathbf{w}}_{\gamma'(t)}}'(0), \gamma'(t)\rangle + \langle \gamma''(t), \mathbf{w}\rangle\big)\theta(\mathbf{w})\mathrm{d}\mathbf{w}\mathrm{d}t.$$

For $\mathbf{u}, \mathbf{v} \in T_pM$, $\|\mathbf{u}\| = 1$, define

$$\mathfrak{A}_r(\mathbf{u}) = -\int_{B_r^{n-1}(\mathbf{u})} \langle {J^{\mathbf{w}}_{\mathbf{u}}}'(0), \mathbf{u}\rangle\theta(\mathbf{w})\mathrm{d}\mathbf{w},$$

$$\mathfrak{B}_r(\mathbf{u}, \mathbf{v}) = -\int_{B_r^{n-1}(\mathbf{u})} \langle \mathbf{v}, \mathbf{w}\rangle\theta(\mathbf{w})\mathrm{d}\mathbf{w}.$$

Corollary

A Riemannian manifold has the tube property if and only if

- $\mathfrak{A}_r(\mathbf{u}) = V(r)$ *does not depend on* $\mathbf{u}$, *and*
- $\mathfrak{B}_r(\mathbf{u}, \mathbf{v})$ *does not depend on* $\mathbf{u}$ *and* $\mathbf{v}$. *In particular,*

$$\mathfrak{B}_r(\mathbf{u}, \mathbf{v}) = \mathfrak{B}_r(\mathbf{u}, \mathbf{0}) \equiv 0.$$

Vanishing of the Curvature Dependent Term

Consider the condition

$$\mathfrak{B}_r(\mathbf{u},\mathbf{v}) = -\int_{B_r^{n-1}(\mathbf{u})} \langle \mathbf{v},\mathbf{w}\rangle \theta(\mathbf{w})\mathrm{d}\mathbf{w} \equiv 0.$$

Definition

The Funk transform $\mathcal{F}\colon \mathcal{C}^\infty(\mathbf{S}^n) \to \mathcal{C}^\infty(\mathbf{S}^n)$ is the integral transform defined by

$$\big(\mathcal{F}(f)\big)(\mathbf{u}) = \int_{\mathbf{S}^n \cap \mathbf{u}^\perp} f(\mathbf{w})\mathrm{d}\mathbf{w}.$$

Theorem (Funk)

$$\mathcal{F}(f) = 0 \iff f \text{ is odd.}$$

Definition

A Riemannian manifold is a D'Atri space if local geodesic symmetries are volume-preserving, or equivalently, $\theta(\mathbf{v}) \equiv \theta(-\mathbf{v})$.

Theorem

A Riemannian manifold has the tube property if and only if it is a D'Atri space and has the tube property for geodesic curves.

Tube Property in Harmonic Spaces

Observation

Harmonic spaces are D'Atri spaces, thus, to check the tube property for a harmonic space, it is enough to check the tube property for geodesics.

Theorem (Z.I. Szabó)

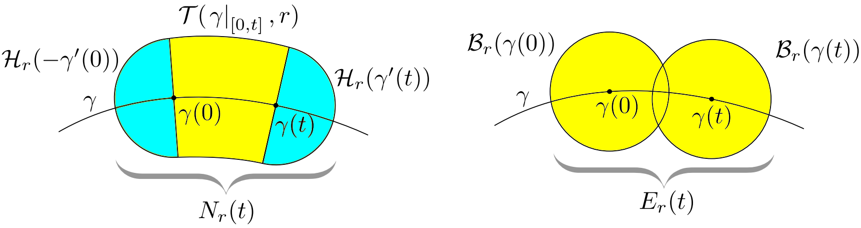
The volume of the intersection of two geodesic balls in a harmonic manifold depends only on the distance between the centers and the radii.

$\Longrightarrow$ The volume of small geodesic balls depends only on the radius.

$\Longrightarrow$ The volume of small geodesic half-balls depends only on the radius.

$\Longrightarrow$ The volume of the union of two geodesic balls depends only on the distance between the centers and the radii.

Tube Property in Harmonic Spaces



For a unit speed geodesic $\gamma\colon [0,l] \to M$ with $\gamma'(0) = \mathbf{u}$, consider the sets

$$N_r(t) = \bigcup_{\tau \in [0,t]} \mathcal{B}_r(\gamma(\tau)) \quad \text{and} \quad E_r(t) = \mathcal{B}_r(\gamma(0)) \cup \mathcal{B}_r(\gamma(t)).$$

We have $|\mathrm{Vol}(N_r(t)) - \mathrm{Vol}(E_r(t))| = O(t^2)$ and therefore,

$$\mathfrak{A}_r(\mathbf{u}) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}(\mathcal{T}(\gamma|_{[0,t]}, r))\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}(N_r(t))\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}(E_r(t))\Big|_{t=0}$$

As $\mathrm{Vol}(E_r(t))$ depends only on r, t, every harmonic space has the tube property.

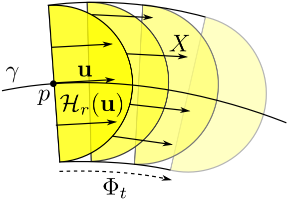
Consequences of the Tube Property

Theorem (P. Günther, F. Prüfer)

Every D'Atri space is ball-homogeneous.

Corollary

In a space having the tube property, the volume of a half-ball depends only on the radius of the half-ball.



- For a unit tangent vector $\mathbf{u} \in T_pM$, consider the geodesic curve γ with $\gamma(0) = p$, $\gamma'(0) = \mathbf{u}$.
- Let $\Pi_t \colon T_pM \to T_{\gamma(t)}M$ be the parallel transport along γ.

- Consider the isotopy $\Phi_t = \exp_{\gamma(t)} \circ \Pi_t \circ \exp_p^{-1}$ defined on a small neighborhood of p for small values of t.
- Let $X(q) = \frac{d}{dt}\Phi_t(q)|_{t=0}$ be the initial velocity vector field of the isotopy.
- For a small r, consider the half-ball $\mathcal{H}_r(\mathbf{u}) = \exp_p(\{\mathbf{v} \in T_pM : \|\mathbf{v}\| \le r, \langle \mathbf{v}, \mathbf{u} \rangle \ge 0\})$.
- For all small t, we have $\Phi_t(\mathcal{H}_r(\mathbf{u})) = \mathcal{H}_r(\gamma'(t))$.

Corollary

If $\mathbf{N}$ is the outer unit normal field of the boundary of $\mathcal{H}_r(\mathbf{u})$ then

$$\int_{\partial \mathcal{H}_r(\mathbf{u})} \langle X, \mathbf{N} \rangle d\sigma = 0.$$

- The boundary of $\mathcal{H}_r(\mathbf{u})$ has two smooth components: the exponential image $\mathbf{B}$ of a flat $(n-1)$-ball and the exponential image $\mathbf{S}^+$ of a hemisphere.
- The images $\Phi_t(\mathbf{B})$ of $\mathbf{B}$ under the isotopy sweep out the tube about γ. Thus,
$$V(r) = \int_{\mathbf{B}} \langle X, -\mathbf{N} \rangle d\sigma = \int_{\mathbf{S}^+} \langle X, \mathbf{N} \rangle d\sigma.$$
- Using Gauss Lemma and the fact that the restrictions of X onto radial geodesic curves starting from p are Jacobi fields, we obtain
$$V(r) = \frac{1}{2} \int_{\mathbb{S}_p^{n-1}} |\langle \mathbf{u}, \mathbf{v} \rangle| \theta(r\mathbf{v}) r^{n-1} d\mathbf{v}.$$

Corollary

If a space has the tube property, then the integrals

$$V(r)=\frac{1}{2}\int_{\mathbb{S}_p^{n-1}}|\langle\mathbf{u},\mathbf{v}\rangle|\theta(r\mathbf{v})r^{n-1}d\mathbf{v}$$

do not depend on the unit vector $\mathbf{u}\in T_pM$*.*

Definition

The cosine transform is the integral transform $\mathcal{F}_C\colon\mathcal{C}^\infty(\mathbf{S}^n)\to\mathcal{C}^\infty(\mathbf{S}^n)$ defined by

$$\big(\mathcal{F}_C(g)\big)(\mathbf{u})=\int_{\mathbf{S}^n}|\langle\mathbf{u},\mathbf{v}\rangle|g(\mathbf{v})\mathrm{d}\mathbf{v}.$$

Theorem

$$\mathcal{F}_C(g)=0 \iff g \textit{ is odd.}$$

Lemma

$$\big(\mathcal{F}_C(1)\big)(\mathbf{u})=\int_{\mathbf{S}^n}|\langle\mathbf{u},\mathbf{v}\rangle|\mathrm{d}\mathbf{v}=2\omega_{n-1}.$$

This means that in a space having the tube property, the function

$$g(\mathbf{v}) = \theta(r\mathbf{v}) - \frac{V(r)}{\omega_{n-1} r^{n-1}}$$

is in the kernek of $\mathcal{F}_C$.

Corollary

If a manifold has the tube property, then it is harmonic. Then the volume of a tube of small radius r about a curve of length l can be expressed as $V(r)l$, where $V(r) = \omega_{n-1} r^{n-1} \theta(r)$.
We also have $V(r) = \frac{\omega_{n-1}}{n\omega_n} \mathrm{Vol}_{n-1}(S_r)$, where S_r is a geodesic sphere of radius r.

Tube Property for Geodesics $\Longrightarrow$ Harmonicity

- We write $\theta(r\mathbf{u}) = \sum_{k=0}^{\infty} a_k(\mathbf{u}) r^k$, where $a_k \colon SM \to \mathbb{R}$.
- $a_0 \equiv 1, \quad a_1 \equiv 0, \quad a_k(-\mathbf{u}) = (-1)^k a_k(\mathbf{u})$.
- Harmonicity $\iff$ a_k is constant for all k, $\Longrightarrow a_{2k+1} \equiv 0$ for all k.
- D'Atri $\iff$ $a_{2k+1} \equiv 0$ for all k.
- We prove by induction on k that if a Riemannian manifold has the tube property for geodesics, then a_i is constant for $0 \le i \le 2k$.

Theorem (L. Vanhecke)

If in a Riemannian manifold a_i is constant for $0 \le i \le 2k$, then $a_{2k+1} \equiv 0$.

Theorem

If a Riemannian manifold has the tube property for geodesics and is a D'Atri space up to order $2k+1$, then it is harmonic up to order $2k+2$.

A Steiner-type Formula

Theorem (E. Abbena, A. Gray, and L. Vanhecke)

$$
\begin{aligned}
V_\gamma(r+\Delta) = & V_\gamma(r) + A_\gamma(r)\Delta - \left(\int_{\mathcal{P}(\gamma,r)} \mu^P(p)dp\right)\frac{\Delta^2}{2} + \\
& + \left(\int_{\mathcal{P}(\gamma,r)} \big(\rho(N(p)) + \tau^P(p) - \tau(p)\big)dp\right)\frac{\Delta^3}{6} + O(\Delta^4),
\end{aligned}
$$

where

- $\mathcal{P}(\gamma, r)$ *is the tubular hypersurface of radius* r *about* γ*;*
- μ^P *is the sum of the principal curvatures of* $\mathcal{P}(\gamma, r)$ *with respect to the outer unit normal* N*;*
- $\rho(N(p)) = \mathrm{Ric}\,(N(p), N(p))$ *is the Ricci curvature of* M *in the direction* $N(p)$*;*
- τ *and* τ^P *are the scalar curvatures of* M *and* $\mathcal{P}(\gamma, r)$*, respectively.*

Corollary

For a connected Riemannian manifold M, the following properties are equivalent:

- M *is harmonic.*
- *For any (geodesic) curve γ, the volume of the tubular hypersurface $\mathcal{P}(\gamma, r)$ depends only on r and the length l_γ of γ.*
- *For any (geodesic) curve γ, the total mean curvature of $\mathcal{P}(\gamma, r)$ depends only on r and l_γ.*
- *For any (geodesic) curve γ, the total scalar curvature of $\mathcal{P}(\gamma, r)$ depends only on r and l_γ (if $\dim M \geq 4$).*

Köszönöm a figyelmet!