On the multiplicity of arrangements of equal zones on the sphere

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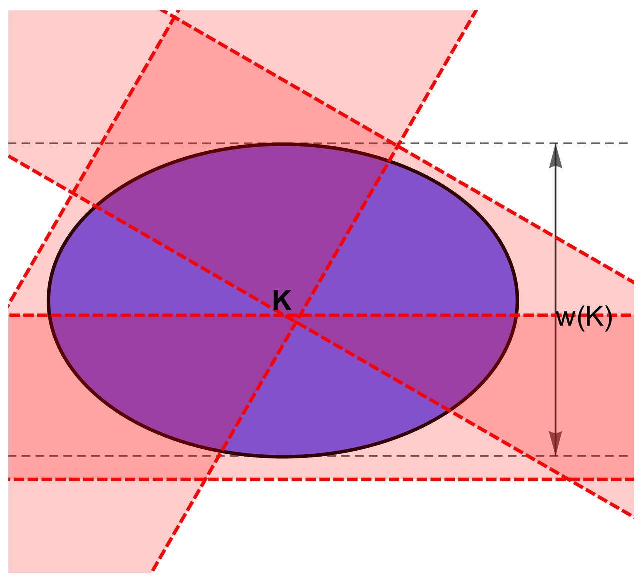
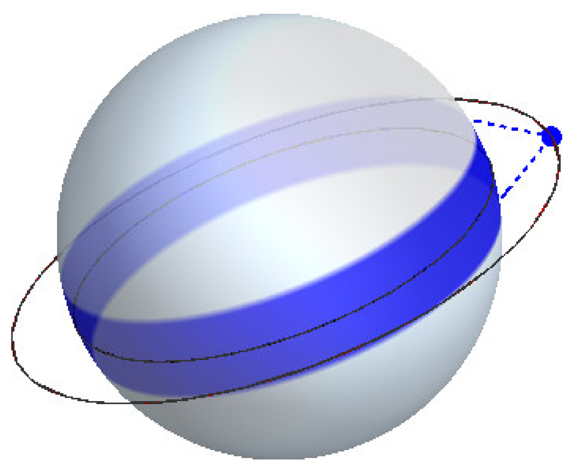
Tarski's Plank Problem (1932)



- A plank of width w in $\mathbb{R}^d$ is the closed region between two parallel hyperplanes at distance w from each other.
- Tarski [6] conjectured that the total width of finitely many planks that cover a convex body K is at least its minimal width $w(K)$. The conjecture was proved by Bang (1950, 1951).
- The affine version of the conjecture (Bang, 1951) that the sum of the relative widths of the planks (w divided by the width of K perpendicular to the plank) is at least 1 is still open. The case of centrally symmetric K was proved by Ball (1991).

L. Fejes Tóth's Spherical Plank Problem (1973)

- A spherical plank or zone is the intersection of S^{d-1} with a Euclidean plank that is symmetric to the origin.
- L. Fejes Tóth [2] asked: what is the minimum spherical width of n equal planks that can cover S^2?



Satellite seeing a zone

- He conjectured that in the optimal configuration the central great circles of the zones all go through an antipodal pair of points and they are distributed evenly.
- Conjecture proved in all dimensions: Jiang, Polyanskii [5] (2017).

Multiplicity of arrangements

- The multiplicity of an arrangement is the largest integer k such that there is a point contained in k members of the arrangement.
- We seek to minimize the multiplicity for given d and n depending on the common width of the zones.
- A prominent example of this problem type: Erdős and Rogers [1] proved that $\mathbb{R}^d$ can be covered by translates of a given convex body with density less than $d\log d + d\log\log d + 4d$ and with multiplicity at most $e(d\log d + d\log\log d + 4d)$.
- For $n \geq d$, the multiplicity of any arrangement with n zones is at least $d-1$ and at most n. In the optimal covering configuration the multiplicity is exactly n, that is, maximal.

Upper bound for arrangements

- Let $\alpha : \mathbb{N} \to (0,1]$ and $k : \mathbb{N} \to \mathbb{N}$ be functions with $\lim_{n\to\infty}\alpha(n) = 0$ such that, with a suitable constant C_d^*

$$\limsup_{n\to\infty} \frac{1}{\alpha(n)^{d-1}} \left(\frac{e\, C_d^*\, n\, \alpha(n)}{k(n)}\right)^{k(n)} = \beta < 1.$$

Theorem. [3] *For each positive integer $d \geq 3$ and function $\alpha(n)$, for sufficiently large n, there exists an arrangement of n zones of spherical half-width $m_d\alpha(n)$ on S^{d-1} such that no point of S^{d-1} belongs to more than $k(n)$ zones, where $m_d = \sqrt{2\pi d} + 1$.*

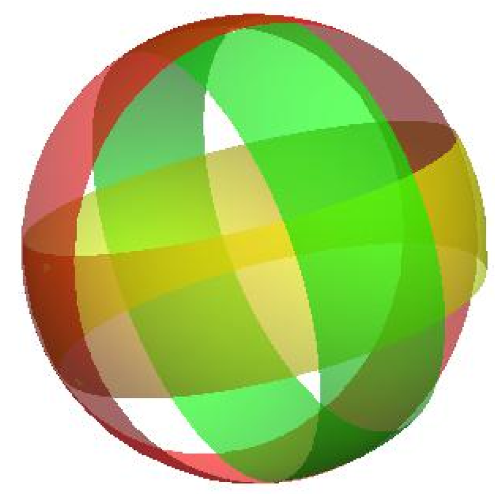
Arrangements with low multiplicity

- The proof of the theorem is probabilistic and it also uses a discretization argument (saturated point sets) on S^{d-1}. The following statements are corollaries.
- For each $d \geq 3$, there exists a positive constant A_d such that for sufficiently large n, there is a covering of S^{d-1} by n zones of half-width $m_d\frac{\ln n}{n}$ with multiplicity at most $A_d \ln n$. (Proved by Frankl, Nagy and Naszódi [4] in the special case $d = 3$.)
- If $\alpha(n) = \frac{1}{n^{1+\delta}}$, $(\delta > 0)$ then $k(n) = const(d, \delta)$. Moreover, if $\delta > d-1$ then $k = d$.

Lower bound for coverings

Theorem. [3] *Let $n \geq 1$ be an integer, and let S^2 be covered by the union of n congruent zones. If each point of S^2 belongs to the interior of at most two zones, then $n \leq 3$. If, moreover, $n = 3$, then the three congruent zones are pairwise orthogonal.*

- It is possible to cover S^2 with 4 or 5 equal zones while keeping the open multiplicity (points in the interiors of zones) at 3.
- For 4 zones, consider 3 great circles passing through the North pole distributed evenly, while setting the common width to be equal to the height of the uncovered part. Finally, put the fourth zone to the equator.



Three pairwise orthogonal zones

Conjecture [3]

Let $d \geq 3$, $n \geq 1$, and let S^{d-1} be covered by n congruent zones. If each point of S^{d-1} belongs to the interior of at most $d-1$ zones, then $n \leq d-1$. If, moreover, $n = d$, then the d congruent zones are pairwise orthogonal. (The conjecture is verified for $d \leq 100$.)

Open Questions

Let $C(n, d)$ denote the minimum multiplicity of coverings of S^{d-1} with n zones of equal width.

- Is $C(n, d)$ bounded for fixed values of d?
- For a fixed d, is $C(n, d)$ monotone in n?
- Can $C(n, d)$ and the density be simultaneously bounded from above by a function of d?

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