A harmonikus terek néhány geometriai jellemzése

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Harmonic manifolds

Definition

A Riemannian manifold M is a (locally) harmonic manifold if the following equivalent definitions are fulfilled.

- Each point $p \in M$ has a neighborhood on which the equation $\Delta u = 0$ has a non-constant solution of the form $u(q) = f(d(p,q))$, where $f\colon (0,a) \to \mathbb{R}$ is real analytic on $(0,a)$. [Ruse 1930].
- At each point $p \in M$, the volume density function $\theta_p = \sqrt{\det(g_{ij})}$ written in normal coordinates centered at p is radial, i.e., $\theta_p(\mathbf{v}) = \bar{\theta}(\|\mathbf{v}\|)$ for some function $\bar{\theta}$.
- Every sufficiently small geodesic sphere has constant mean curvature.
- Every sufficiently small geodesic sphere has constant scalar curvature. (if $\dim M > 2$)
- If f is harmonic on a neighborhood of a sufficiently small geodesic ball $B(p,r)$, then

$$f(p) = \frac{1}{\mathrm{Vol}(S(p,r))} \int_{S(p,r)} f \, \mathrm{d}\sigma.$$

Examples of harmonic manifolds

Definition

A metric space (X, d) is called 2 point homogeneous if $\forall p, q, p', q' \in X$ such that $d(p,q) = d(p', q') \Rightarrow$ there is an isometry $\Phi\colon X \to X$ such that $\Phi(p) = p'$ and $\Phi(q) = q'$.

Proposition

A connected Riemannian manifold is

2 point homogeneous $\iff$
- *complete*
- *its isometry group acts transitively on the bundle of unit tangent vectors*

In particular, manifolds locally isometric to a 2 point homogeneous space are harmonic.

Theorem (J. Tits, H.C. Wang, Z.I. Szabó)

Connected 2 point homogeneous Riemannian manifolds are
- *the Euclidean spaces $\mathbf{E}^n$,*
- *the simply connected rank 1 symmetric spaces*

$$\mathbf{S}^n, \mathbb{C}\mathbf{P}^n, \mathbb{H}\mathbf{P}^n, \mathbb{O}\mathbf{P}^2, \quad \mathbf{H}^n, \mathbb{C}\mathbf{H}^n, \mathbb{H}\mathbf{H}^n, \mathbb{O}\mathbf{H}^2,$$

- *the real projective spaces $\mathbb{R}\mathbf{P}^n$.*

Examples of harmonic manifolds

Lichnerowicz Conjecture

Every harmonic manifold is locally isometric to a 2 *point homogeneous space.*

Theorem (Z.I. Szabó)

If a simply connected and connected harmonic manifold is compact, then it is a rank 1 *symmetric space.*

- In the non-compact case, the Lichnerowicz conjecture is false.

Theorem (E. Damek, F. Ricci)

There are many non-symmetric harmonic manifolds among solvable extensions of Heisenberg type 2-step nilpotent Lie groups equipped with left invariant Riemannian metrics.

Theorem (J. Heber)

Every homogeneous harmonic manifold is locally isometric to a 2 *point homogeneous space or a Damek–Ricci space.*

Tubes about a curve

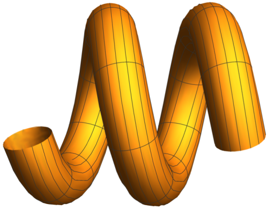
- M – connected Riemannian manifold;
- $\exp\colon \mathring{T}M \to M$ – its exponential map;
- $\gamma\colon [a,b] \to M$ – an injective regular curve;
- For $r > 0$, set

$$T(\gamma, r) = \bigcup_{t\in[a,b]} \{\mathbf{v} \in T_{\gamma(t)}M, \mathbf{v} \perp \gamma'(t), \text{ and } \|\mathbf{v}\| \le r\}.$$

Definition

Assume that r is small enough to guarantee that the exponential map is defined and injective on $T(\gamma, r)$. Then we define the tube of radius r about γ by

$$\mathcal{T}(\gamma, r) = \exp(T(\gamma, r)).$$



The Tube Property

Theorem (Hotelling)

In the Euclidean and spherical spaces, the volume of $\mathcal{T}(\gamma, r)$ depends only on the length of γ and the radius r.

Theorem (H. Weyl)

The volume of a tube of radius r about a submanifold of $\mathbf{E}^n$ or $\mathbf{S}^n$ depends only on intrinsic invariants of the submanifold and on r.

Definition

- We say that a Riemannian manifold has the tube property if there is a function $V\colon [0, \infty) \to \mathbb{R}$ such that

$$\mathrm{Vol}(\mathcal{T}(\gamma, r)) = V(r) l_\gamma \tag{1}$$

  for any smooth injective regular curve γ of length l_γ and any sufficiently small r.
- the manifold has the tube property for geodesics if (1) holds for any injective geodesic arc and any small radius r.

Theorem (A. Gray, L. Vanhecke, 1982)

Every rank 1 symmetric space has the tube property. (In these spaces, the volume of tubes were computed explicitly.)

Main Theorem

For a connected Riemannian manifold, the following properties are equivalent

- *the manifold is harmonic;*
- *the manifold has the tube property;*
- *the manifold has the tube property for geodesic curves.*

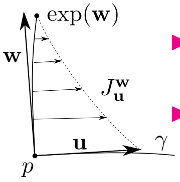
In a harmonic manifold, the volume of a tube of radius r about a curve of length l_γ is

$$\omega_{n-1} r^{n-1} \bar{\theta}(r) l_\gamma = \frac{\omega_{n-1}}{n\omega_n} \mathrm{Vol}_{n-1}(S_r) l_\gamma,$$

where

- *S_r is a geodesic sphere of radius r*
- *ω_{n-1} denotes the $(n-1)$-dimensional volume of the unit sphere in the Euclidean space $\mathbb{E}^n$.*

Some tools of the proof



- For $\mathbf{0} \neq \mathbf{u} \in T_pM$ set $B_r^{n-1}(\mathbf{u}) = \{\mathbf{w} \in T_pM : \mathbf{w} \perp \mathbf{u},\ \|\mathbf{w}\| \leq r\}$.
- For $\mathbf{0} \neq \mathbf{u} \in T_pM$ and $\mathbf{0} \neq \mathbf{w} \in B_r^{n-1}(\mathbf{u})$, let $J_{\mathbf{u}}^{\mathbf{w}}$ be the Jacobi field along the geodesic $t \mapsto \exp(t\mathbf{w})$ defined by

$$J_{\mathbf{u}}^{\mathbf{w}}(0) = \mathbf{u} \quad \text{and} \quad J_{\mathbf{u}}^{\mathbf{w}}(1) = \mathbf{0}.$$

Theorem

The volume of the tube of radius r about the unit speed curve $\gamma\colon [a,b] \to M$ equals

$$-\int_a^b \int_{B_r^{n-1}(\gamma'(t))} \big(\langle {J_{\gamma'(t)}^{\mathbf{w}}}'(0), \gamma'(t)\rangle + \langle \gamma''(t), \mathbf{w}\rangle\big)\theta(\mathbf{w})\mathrm{d}\mathbf{w}\mathrm{d}t.$$

Some tools of the proof

Definition

The Funk transform $\mathcal{F}\colon \mathcal{C}^\infty(\mathbf{S}^n) \to \mathcal{C}^\infty(\mathbf{S}^n)$ is the integral transform defined by

$$\big(\mathcal{F}(f)\big)(\mathbf{u}) = \int_{\mathbf{S}^n \cap \mathbf{u}^\perp} f(\mathbf{w})\,\mathrm{d}\mathbf{w}.$$

Definition

The cosine transform is the integral transform $\mathcal{F}_C\colon \mathcal{C}^\infty(\mathbf{S}^n) \to \mathcal{C}^\infty(\mathbf{S}^n)$ defined by

$$\big(\mathcal{F}_C(f)\big)(\mathbf{u}) = \int_{\mathbf{S}^n} |\langle \mathbf{u}, \mathbf{v}\rangle| f(\mathbf{v})\,\mathrm{d}\mathbf{v}.$$

Theorem

$$\mathcal{F}(f) = 0 \iff f \text{ is odd.}$$
$$\mathcal{F}_C(f) = 0 \iff f \text{ is odd.}$$

Some tools of the proof

Observation

Harmonic spaces are D'Atri spaces, thus, to check the tube property for a harmonic space, it is enough to check the tube property for geodesics.

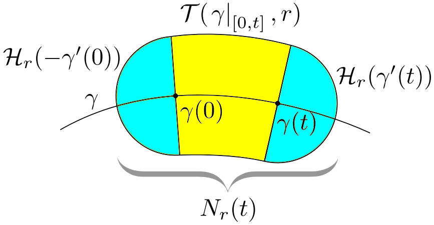
Theorem (Z.I. Szabó)

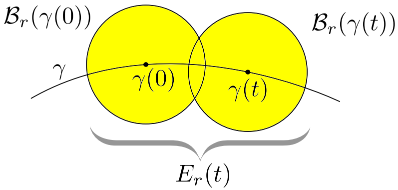
The volume of the intersection of two geodesic balls in a harmonic manifold depends only on the distance between the centers and the radii.

$\Longrightarrow$ The volume of small geodesic balls depends only on the radius.

$\Longrightarrow$ The volume of small geodesic half-balls depends only on the radius.

$\Longrightarrow$ The volume of the union of two geodesic balls depends only on the distance between the centers and the radii.





A Steiner-type formula

Theorem (E. Abbena, A. Gray, and L. Vanhecke)

$$
\begin{aligned}
V_\gamma(r+\Delta) = & V_\gamma(r) + A_\gamma(r)\Delta - \left(\int_{\mathcal{P}(\gamma,r)} \mu^P(p)\,\mathrm{d}p\right)\frac{\Delta^2}{2} \\
& + \left(\int_{\mathcal{P}(\gamma,r)} \big(\rho(N(p)) + \tau^P(p) - \tau(p)\big)\,\mathrm{d}p\right)\frac{\Delta^3}{6} + O(\Delta^4),
\end{aligned}
$$

where

- $V_\gamma(r)$ *is the volume of the tube of radius* r *about* γ*;*
- $\mathcal{P}(\gamma,r)$ *is the tubular hypersurface of radius* r *about* γ*;*
- $A_\gamma(r)$ *is the* $(n-1)$*-dimensional volume of the hypersurface* $\mathcal{P}(\gamma,r)$*;*
- μ^P *is the sum of the principal curvatures of* $\mathcal{P}(\gamma,r)$ *with respect to the outer unit normal* N*;*
- $\rho(N(p)) = \mathrm{Ric}\,(N(p),N(p))$ *is the Ricci curvature of* M *in the direction* $N(p)$*;*
- τ *and* τ^P *are the scalar curvatures of* M *and* $\mathcal{P}(\gamma,r)$*, respectively.*

Corollary

For a connected Riemannian manifold M, the following properties are equivalent:

- M *is harmonic.*
- *For any (geodesic) curve γ, the volume of the tubular hypersurface $\mathcal{P}(\gamma, r)$ depends only on r and the length l_γ of γ.*
- *For any (geodesic) curve γ, the total mean curvature of $\mathcal{P}(\gamma, r)$ depends only on r and l_γ.*
- *For any (geodesic) curve γ, the total scalar curvature of $\mathcal{P}(\gamma, r)$ depends only on r and l_γ (if $\dim M \geq 4$).*

Köszönöm a figyelmet!