Konstans görbületű síkprojektív Randers felületek holonómia csoportja

Muzsnay Zoltán

(Debreceni Egyetem)

Szeged, 2017. október 6.

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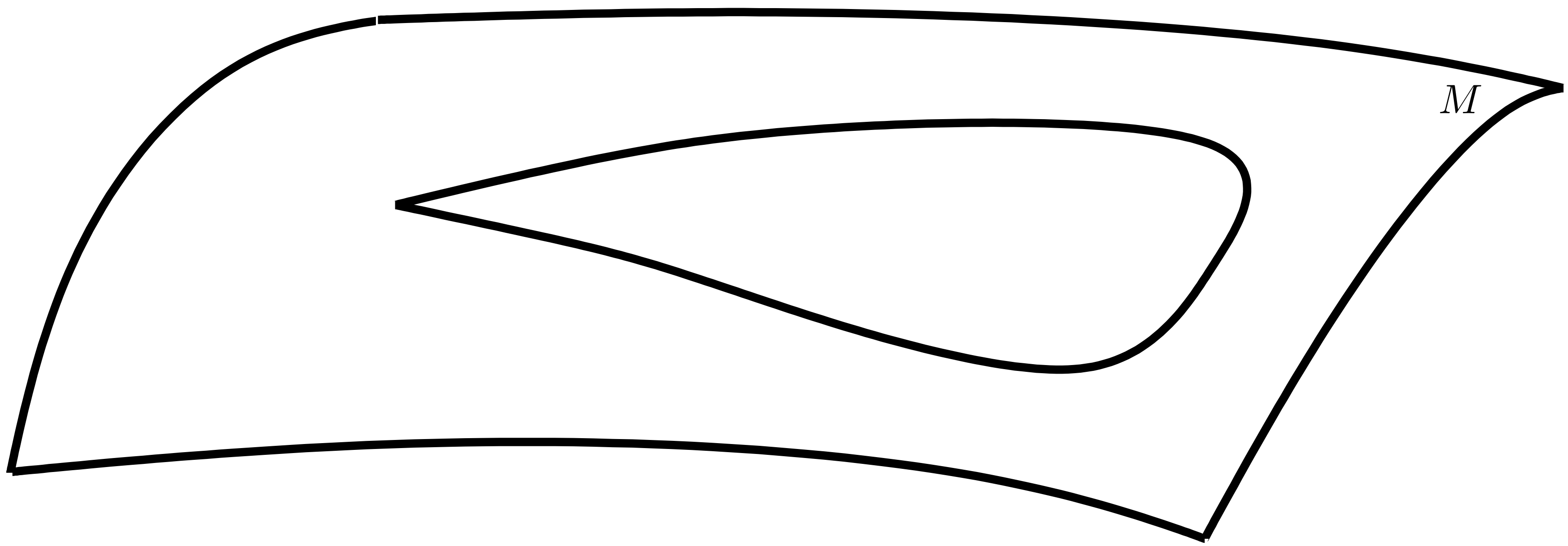

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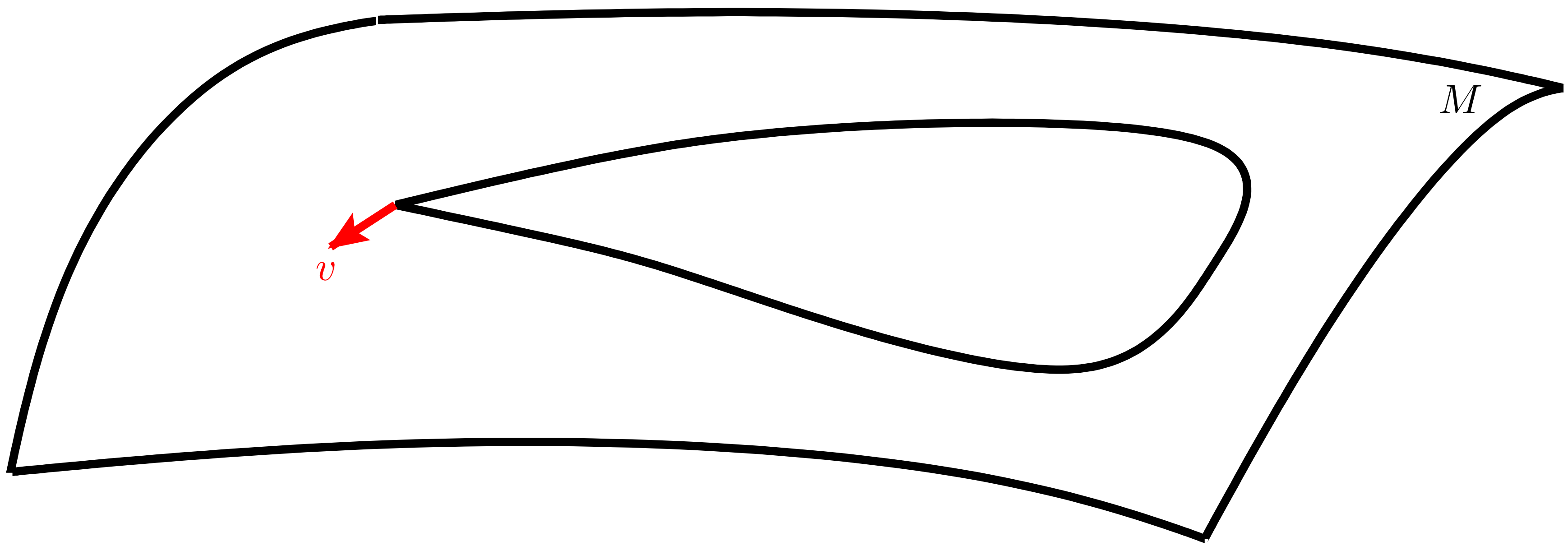

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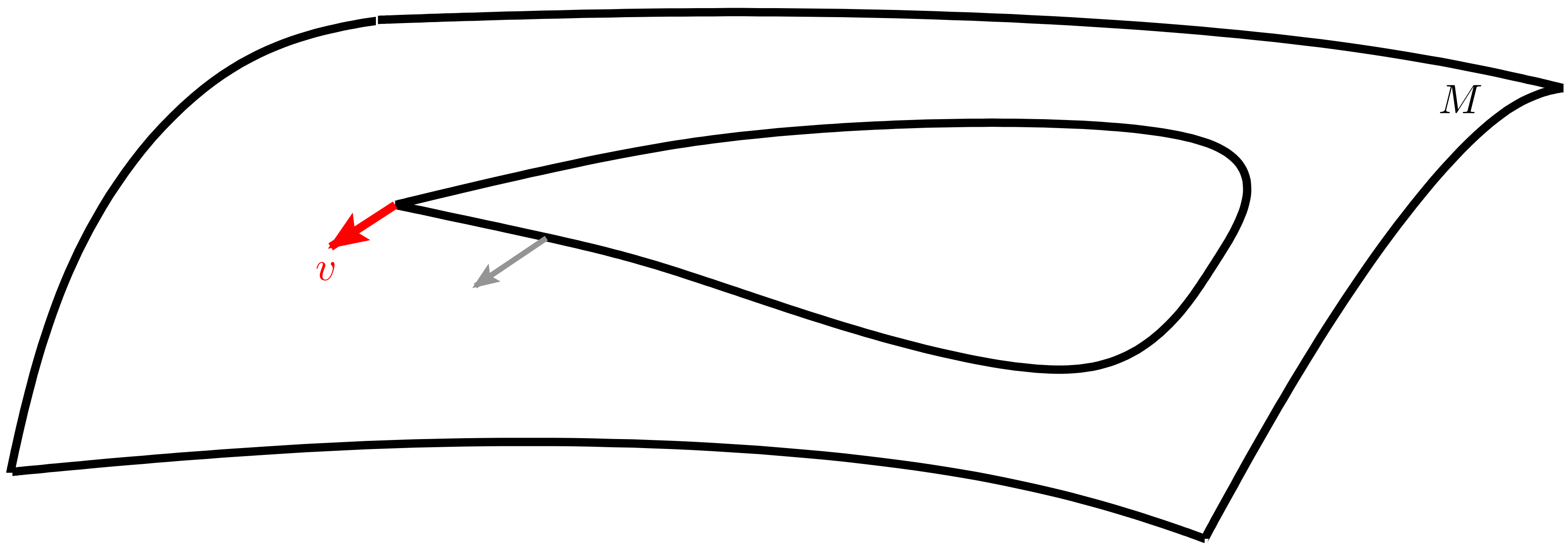

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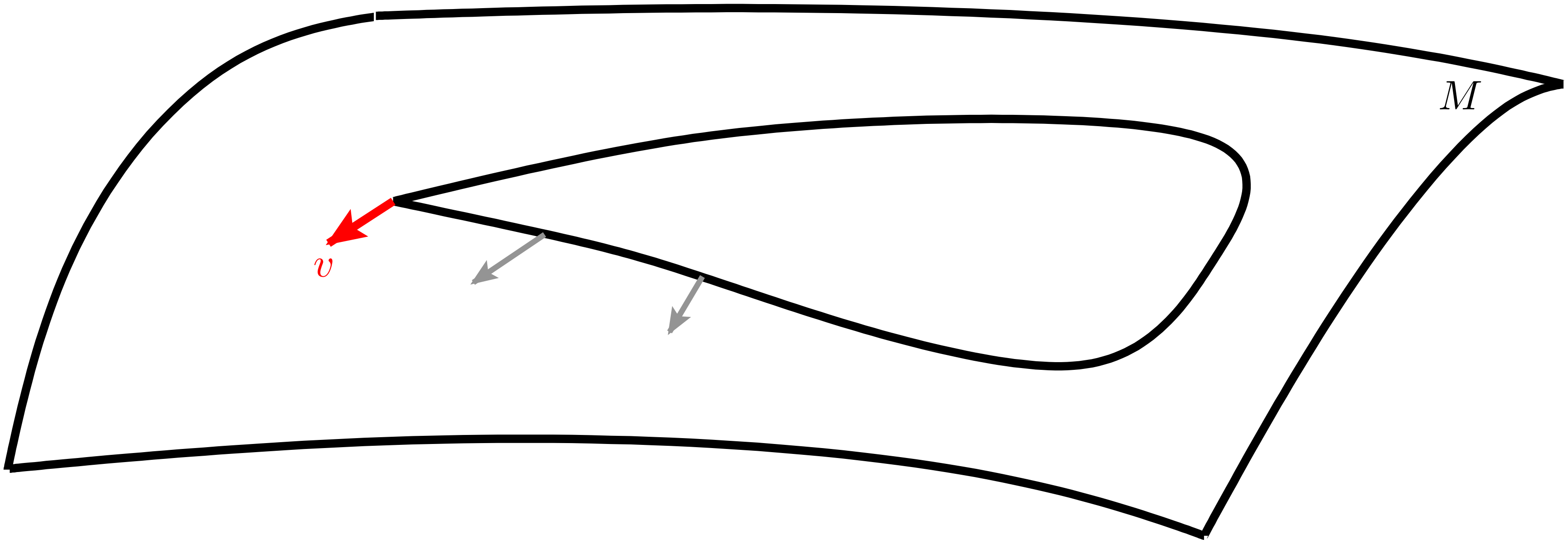

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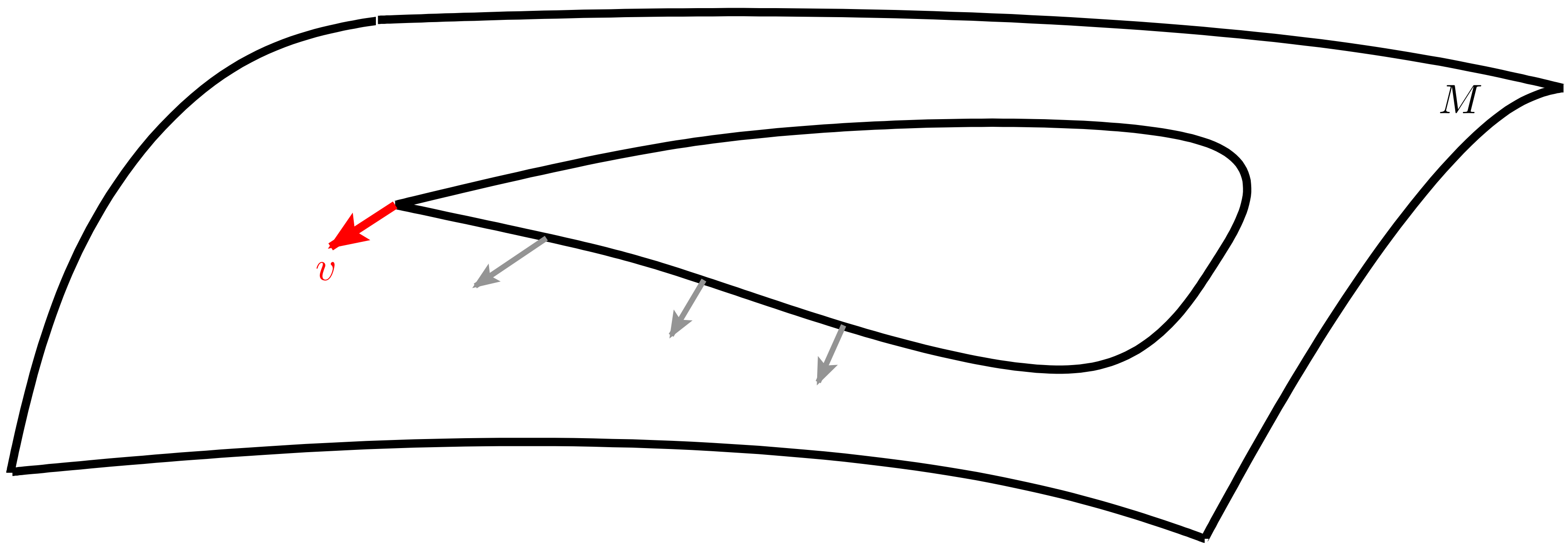

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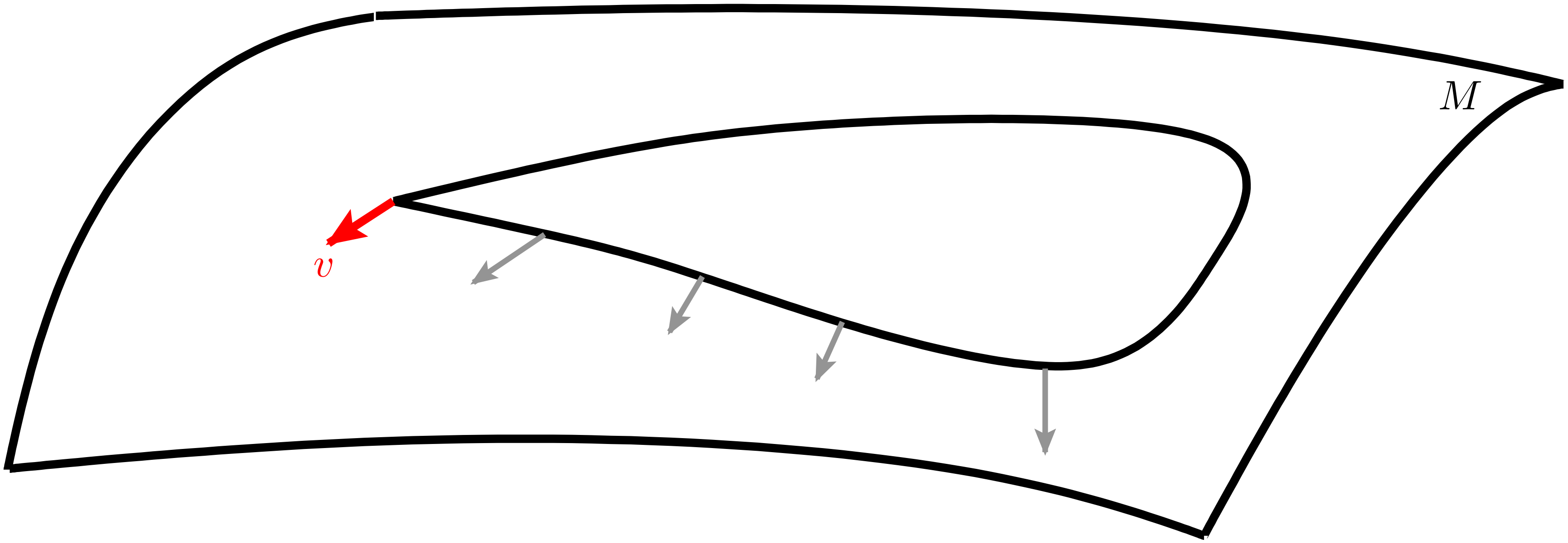

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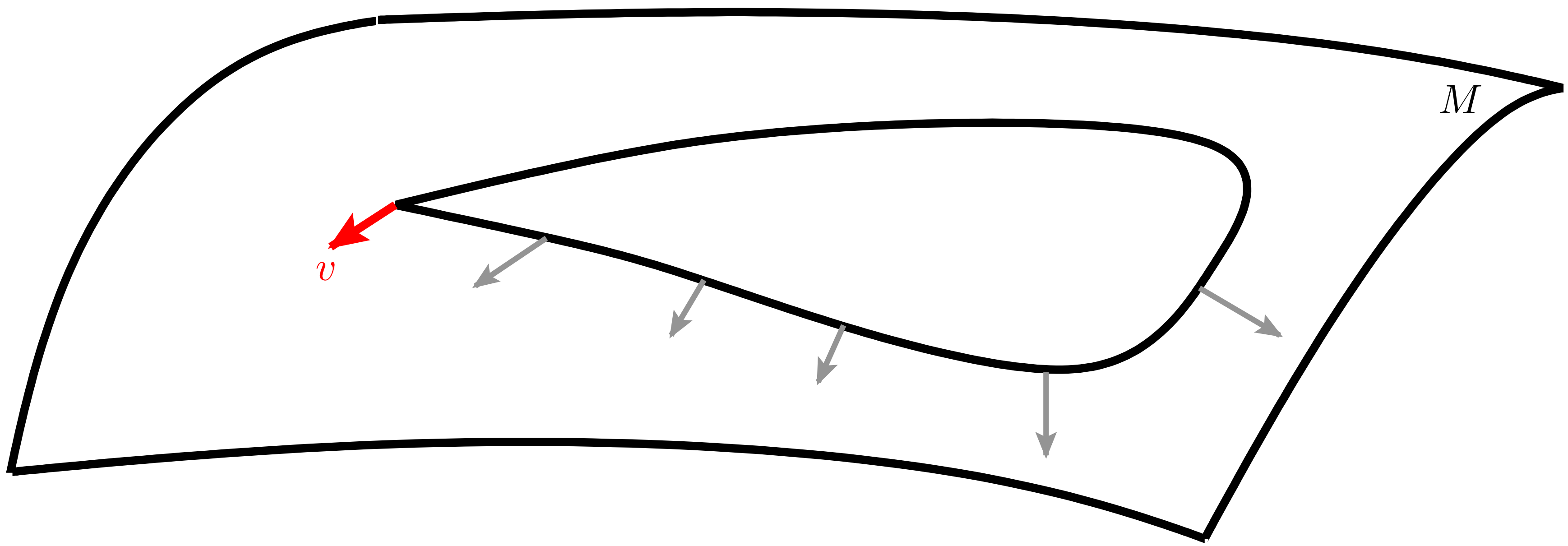

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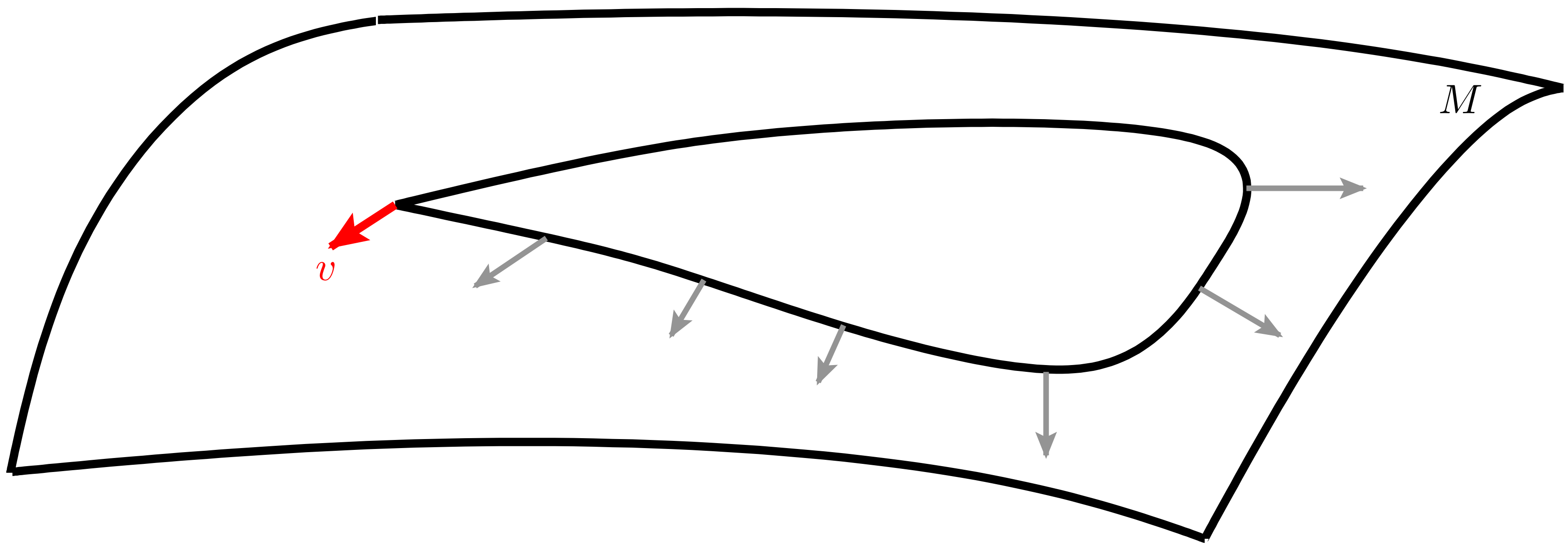

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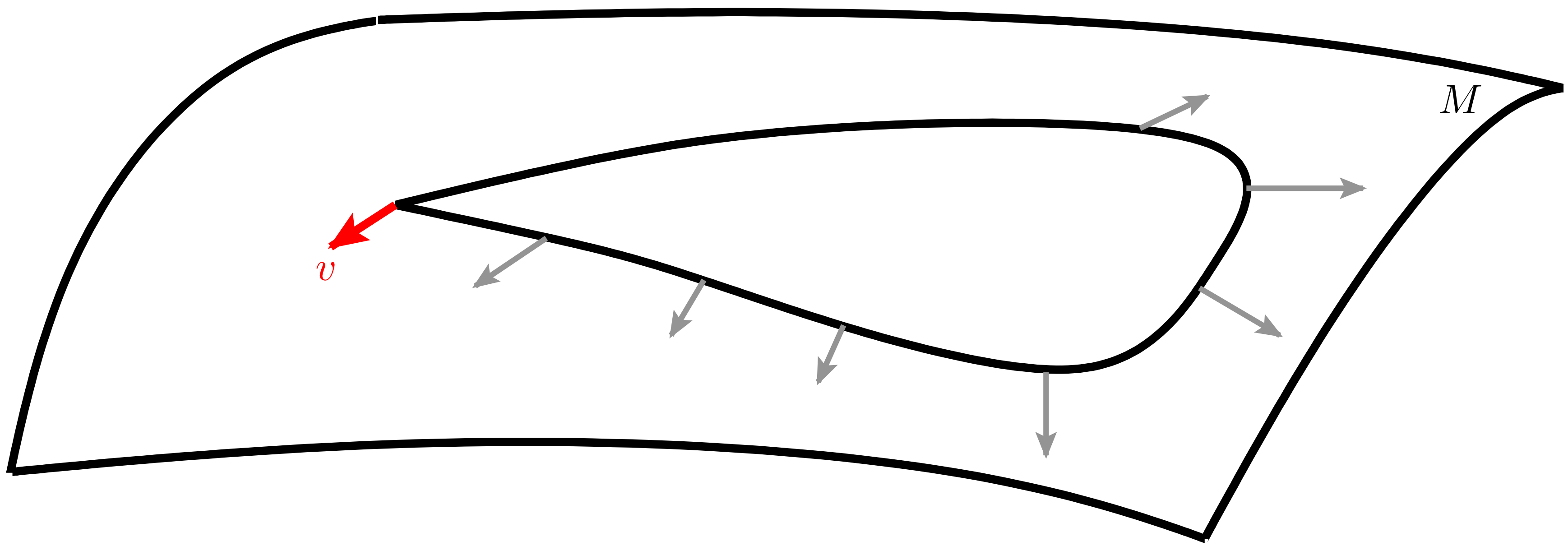

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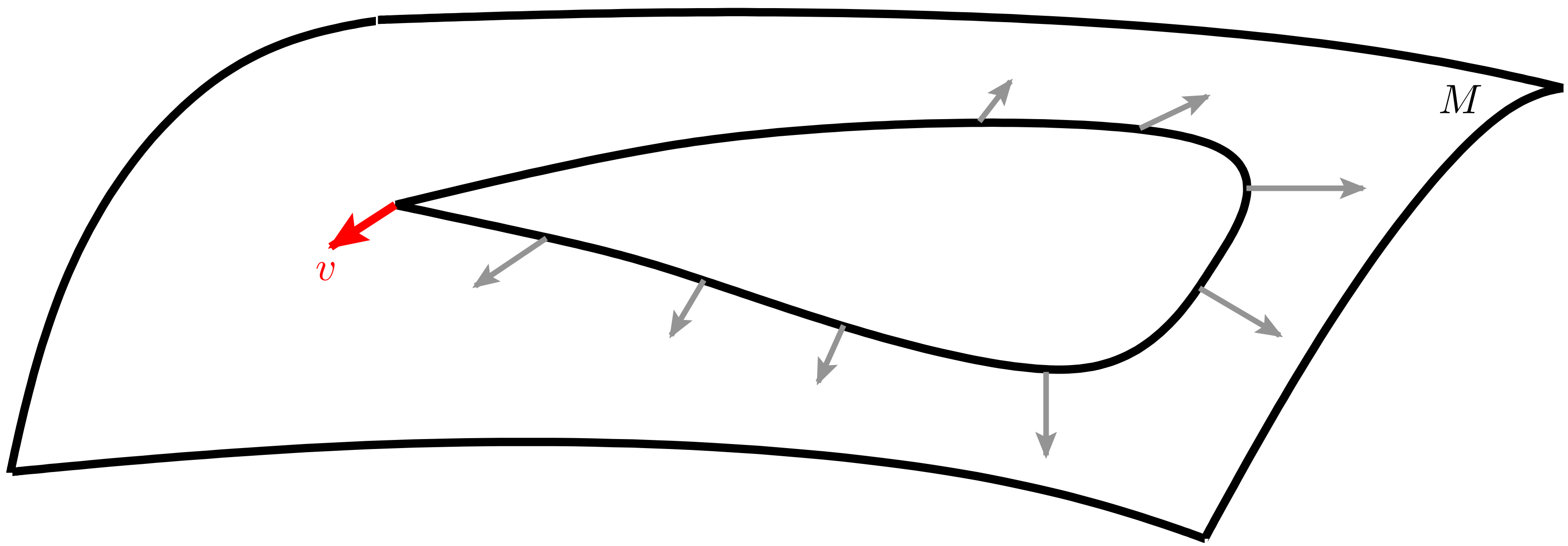

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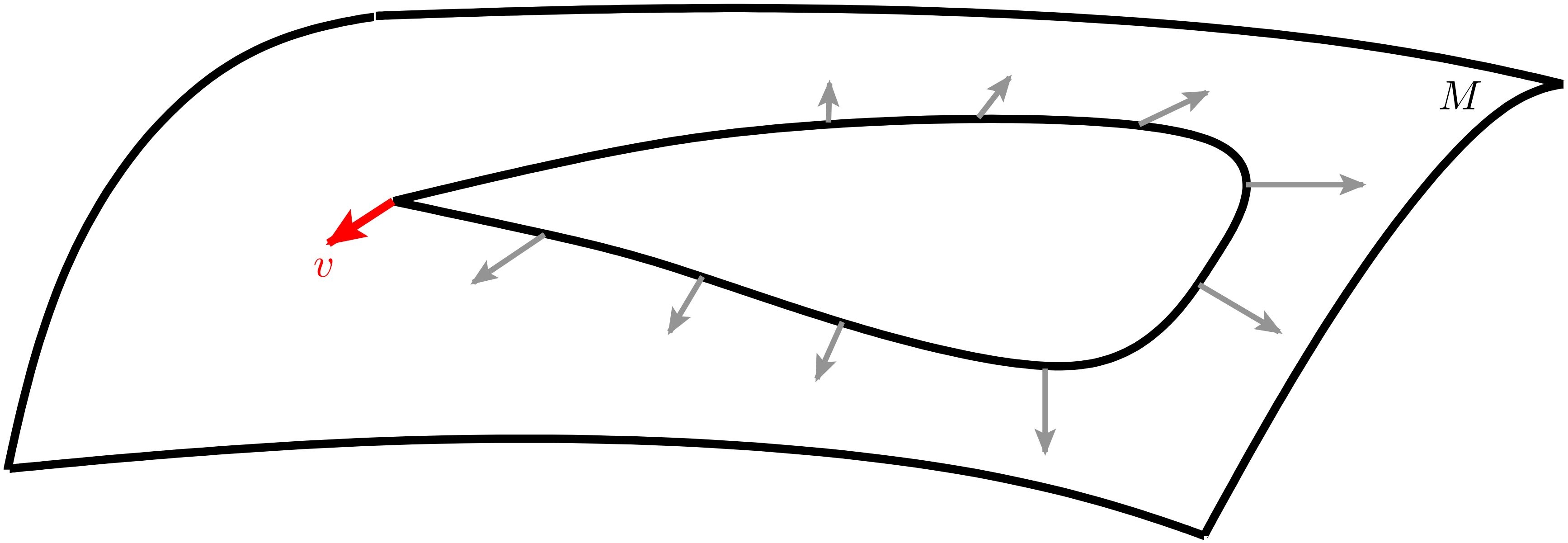

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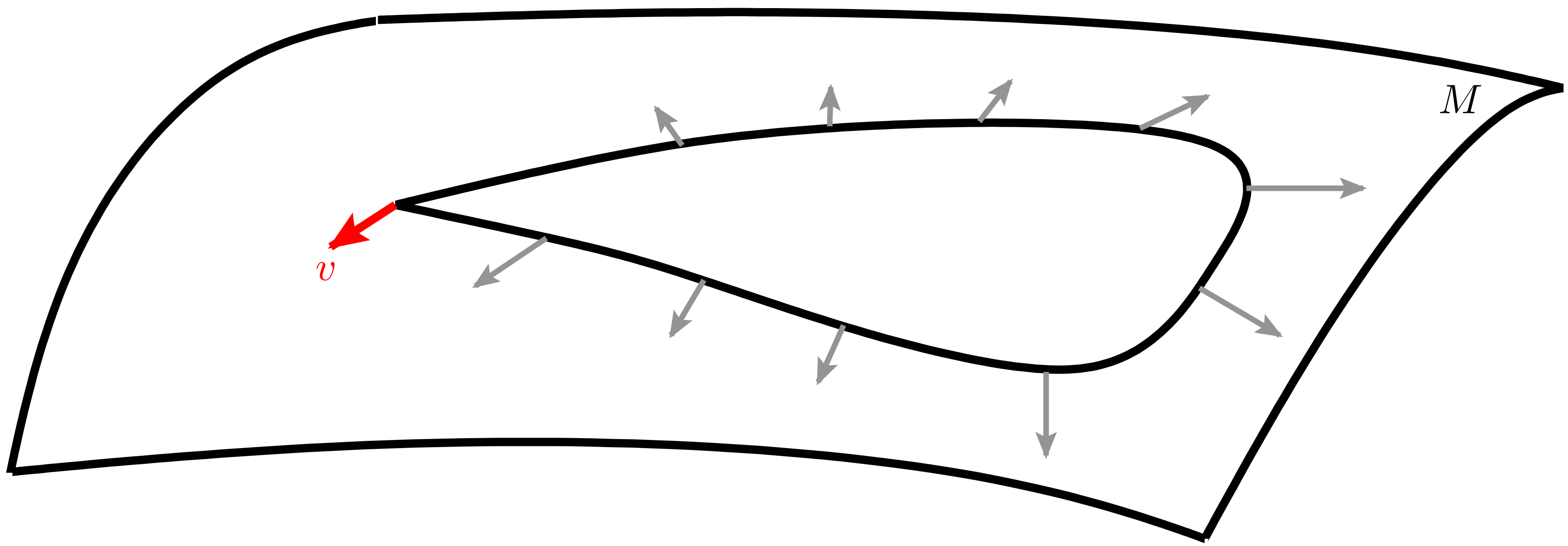

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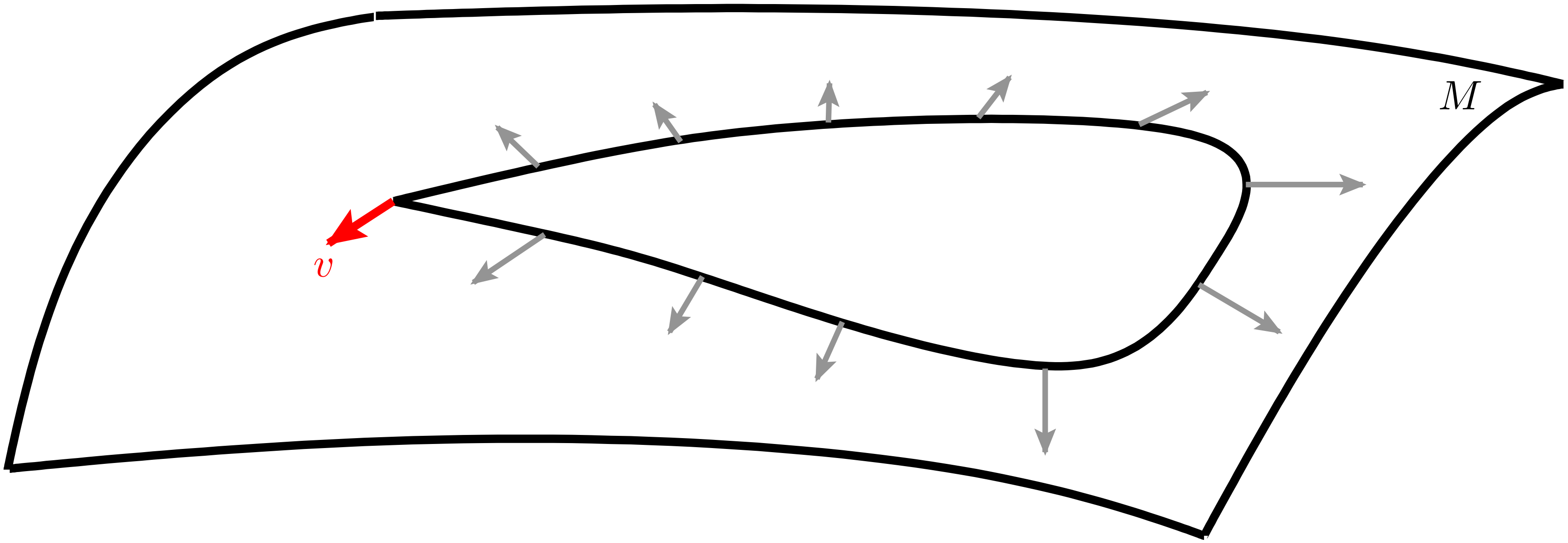

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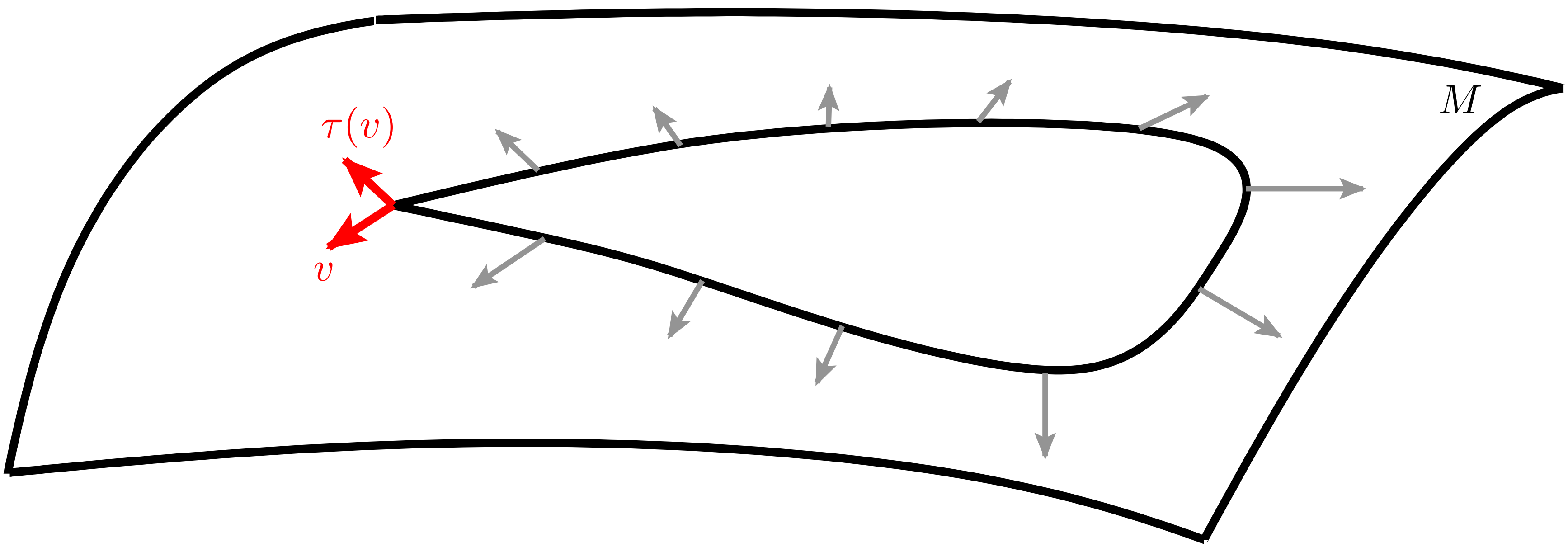

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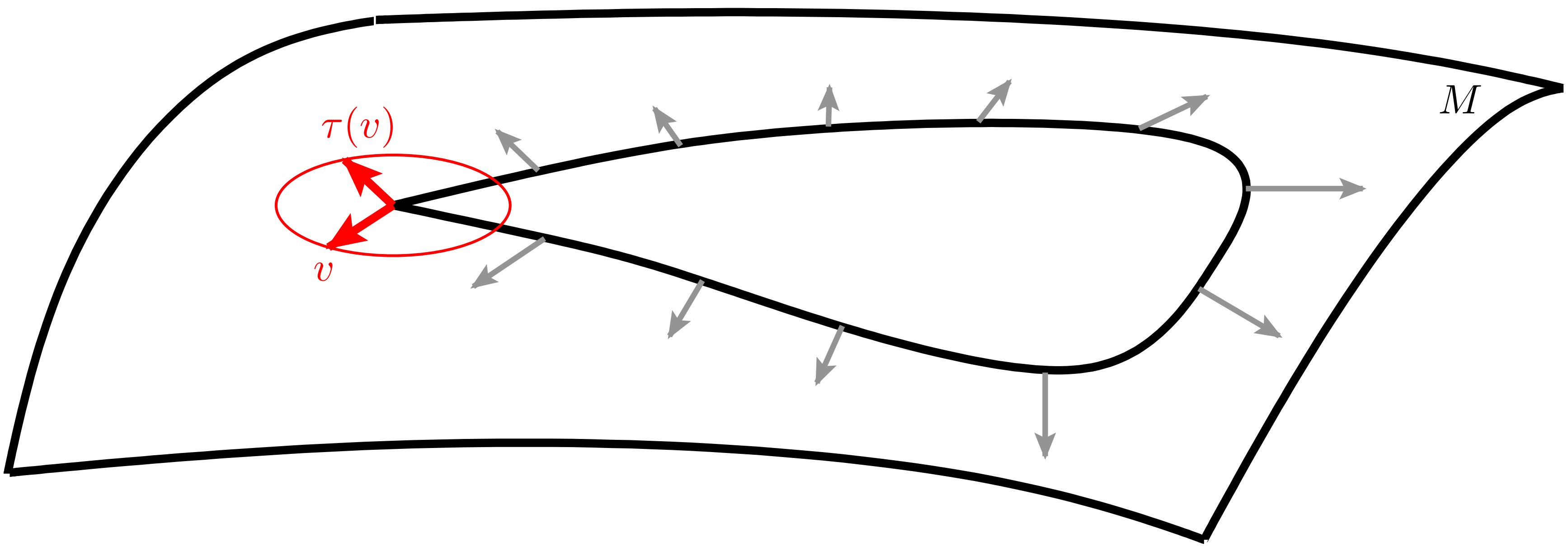

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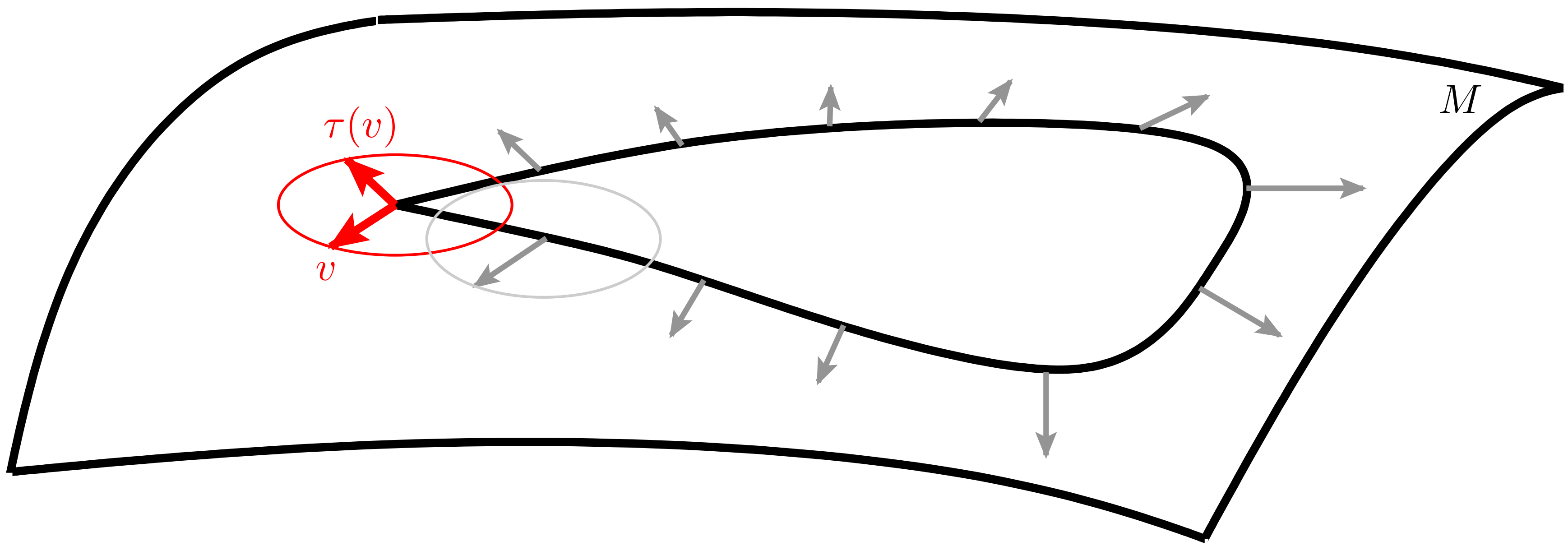

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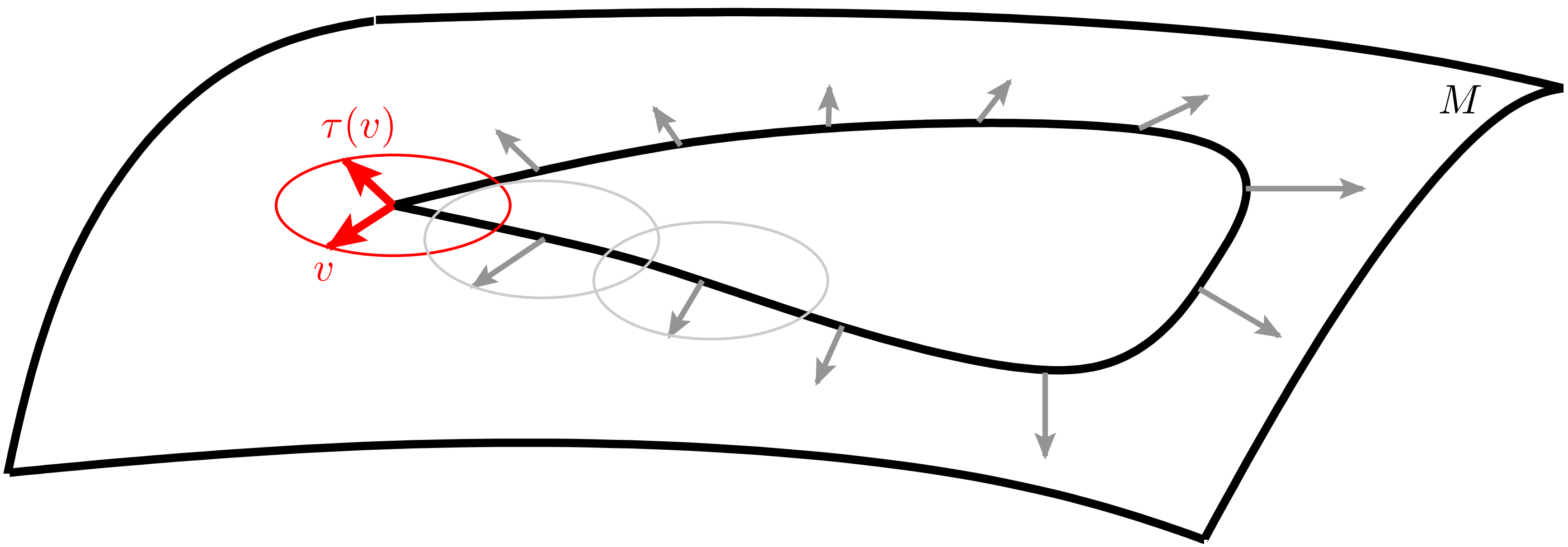

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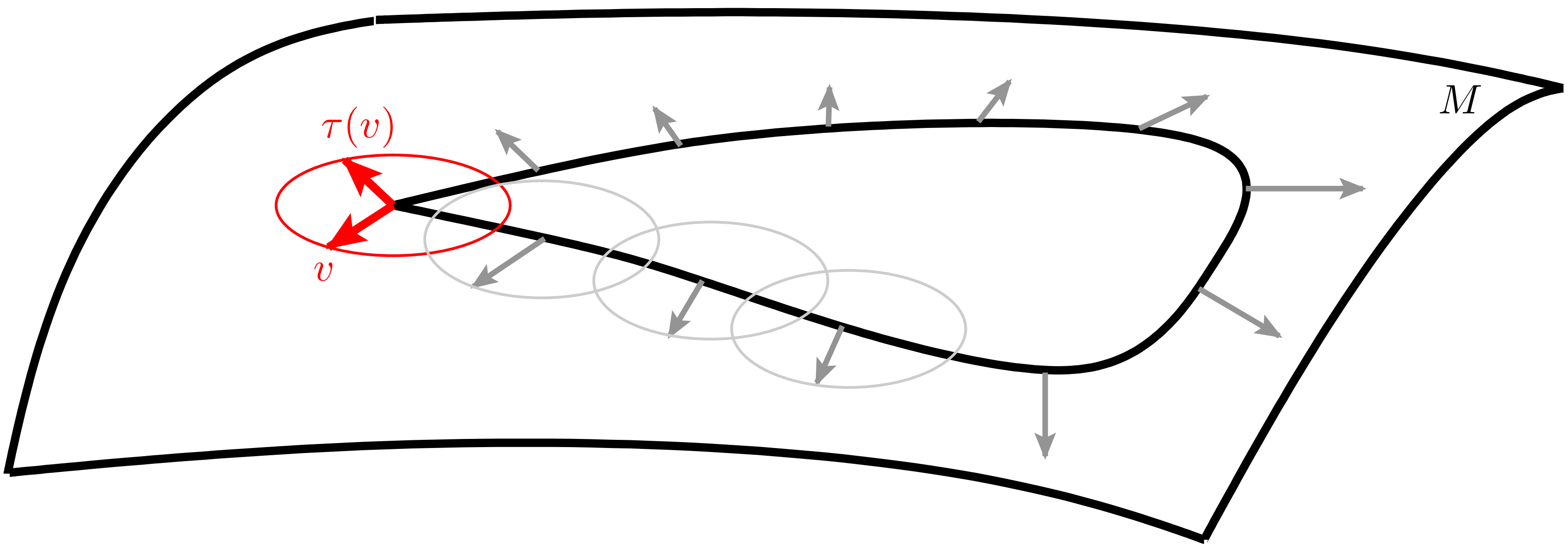

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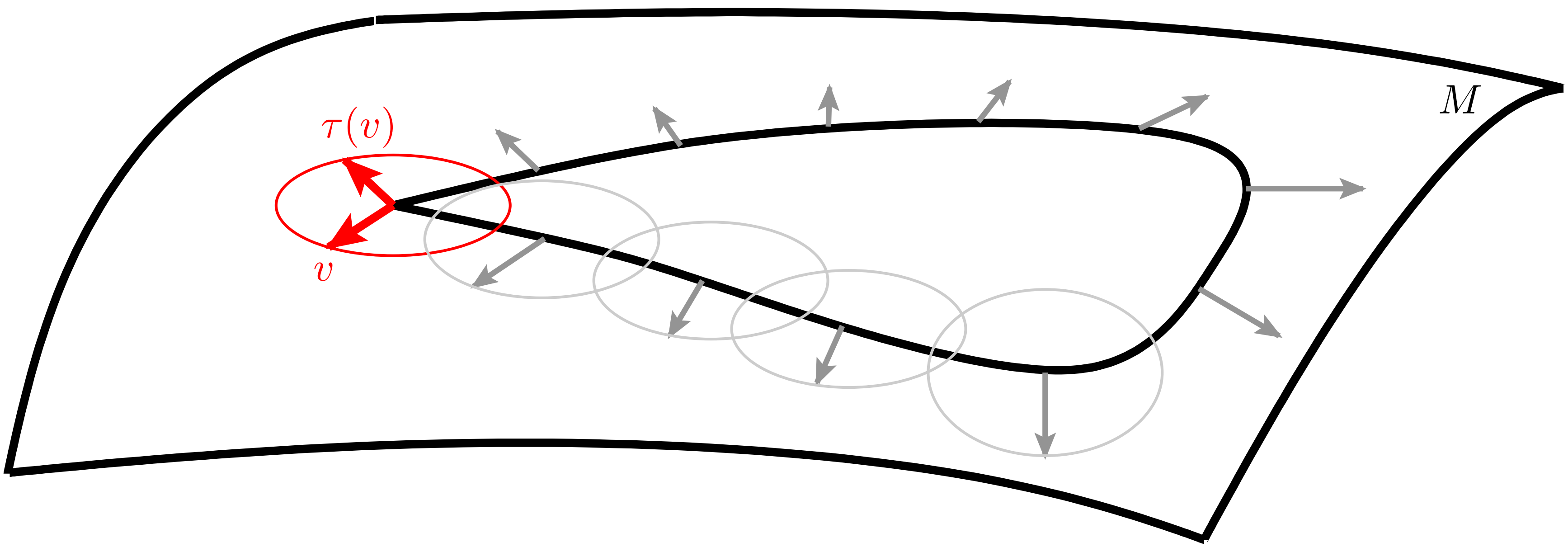

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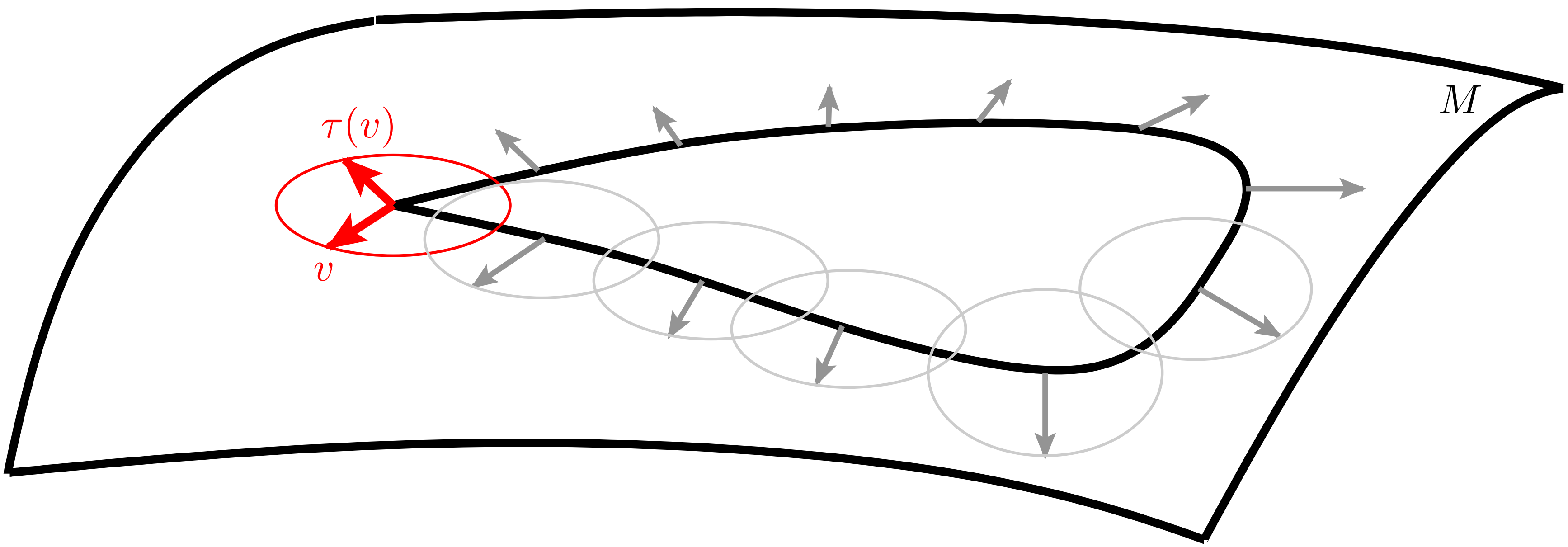

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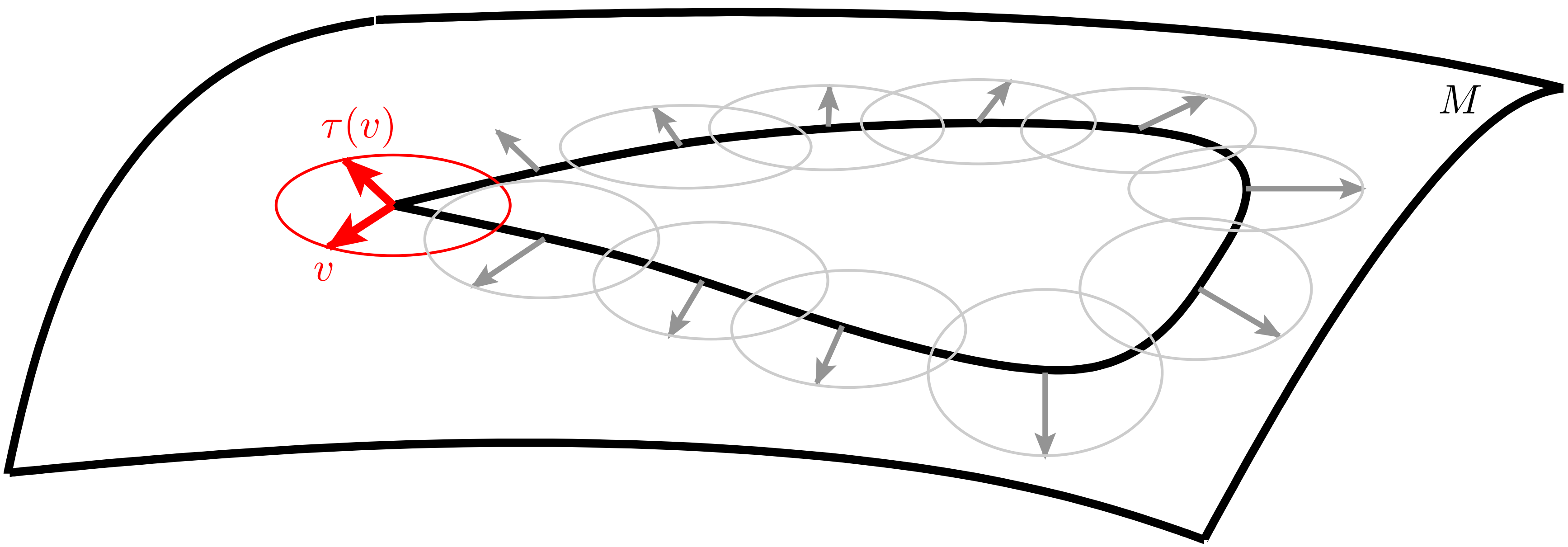

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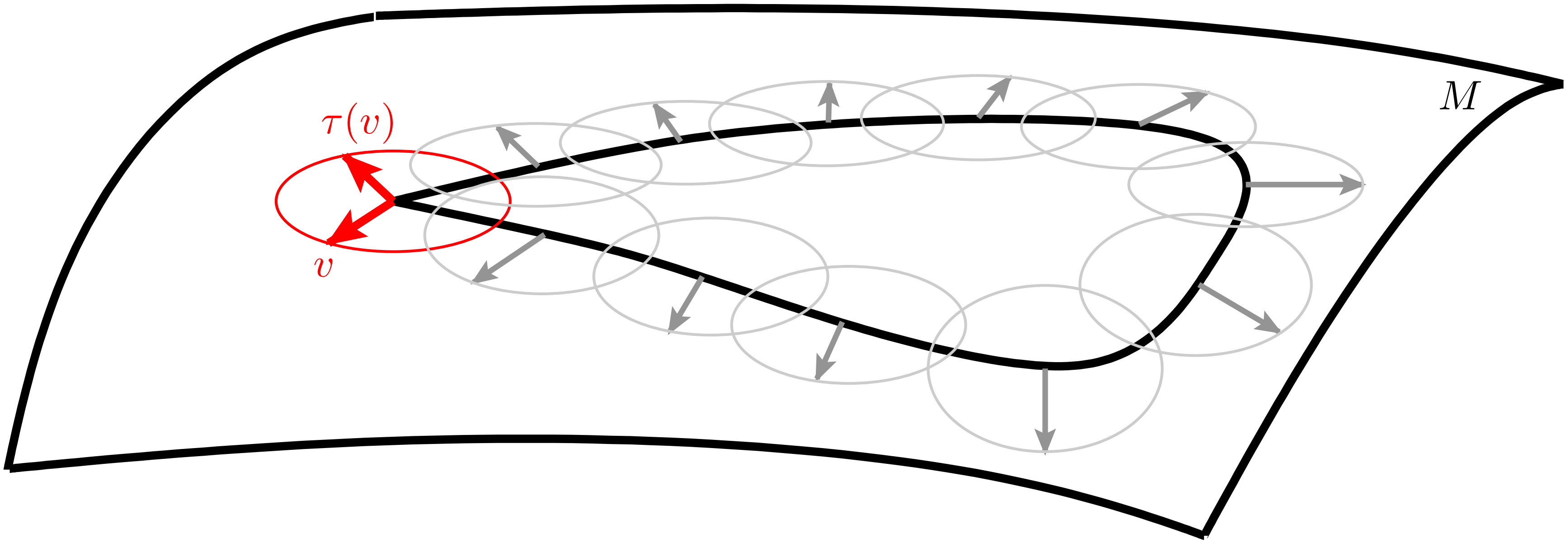

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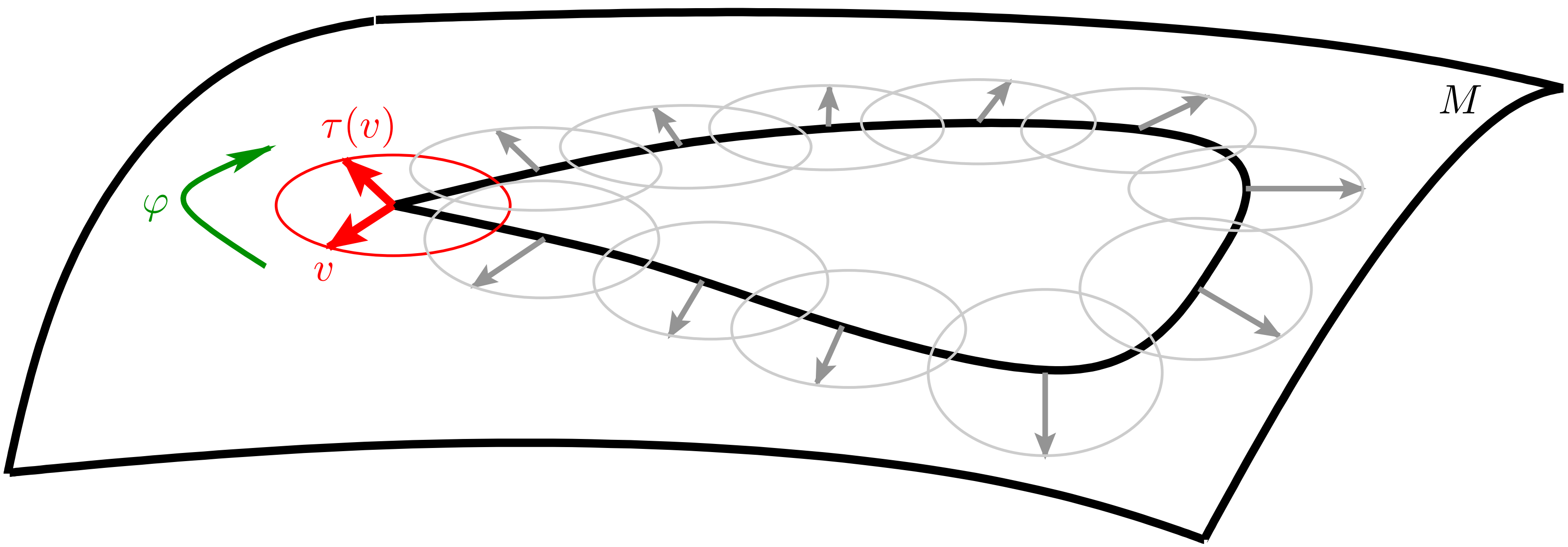

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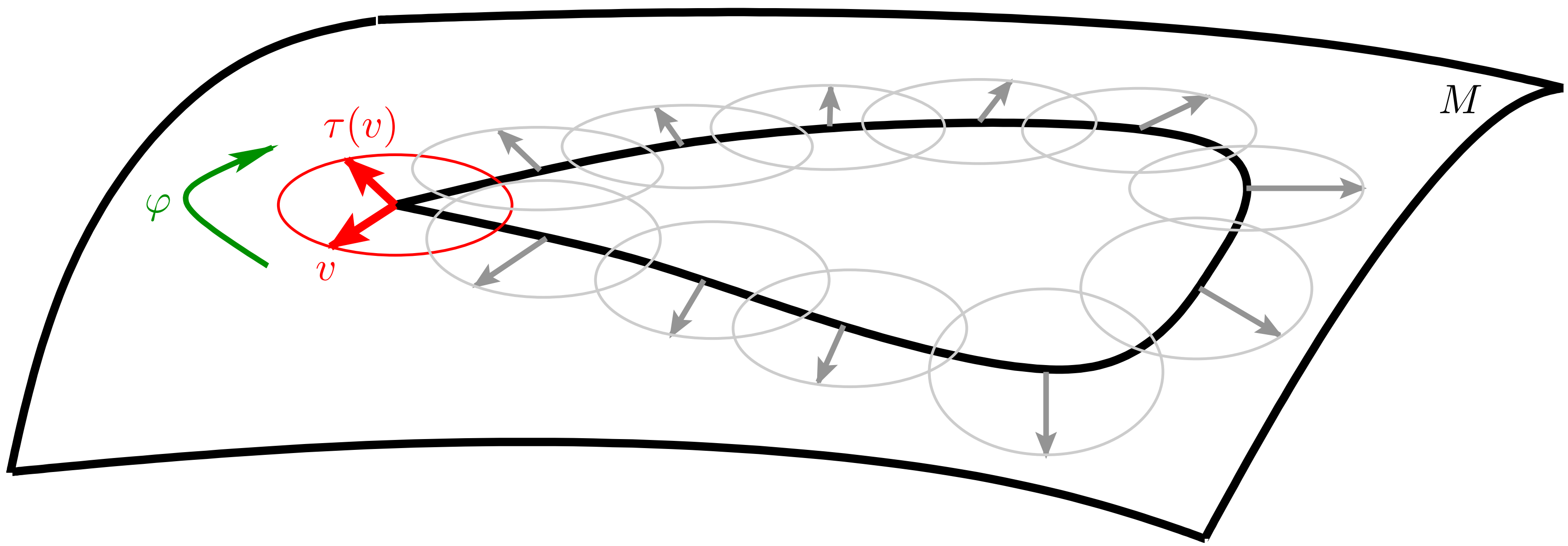

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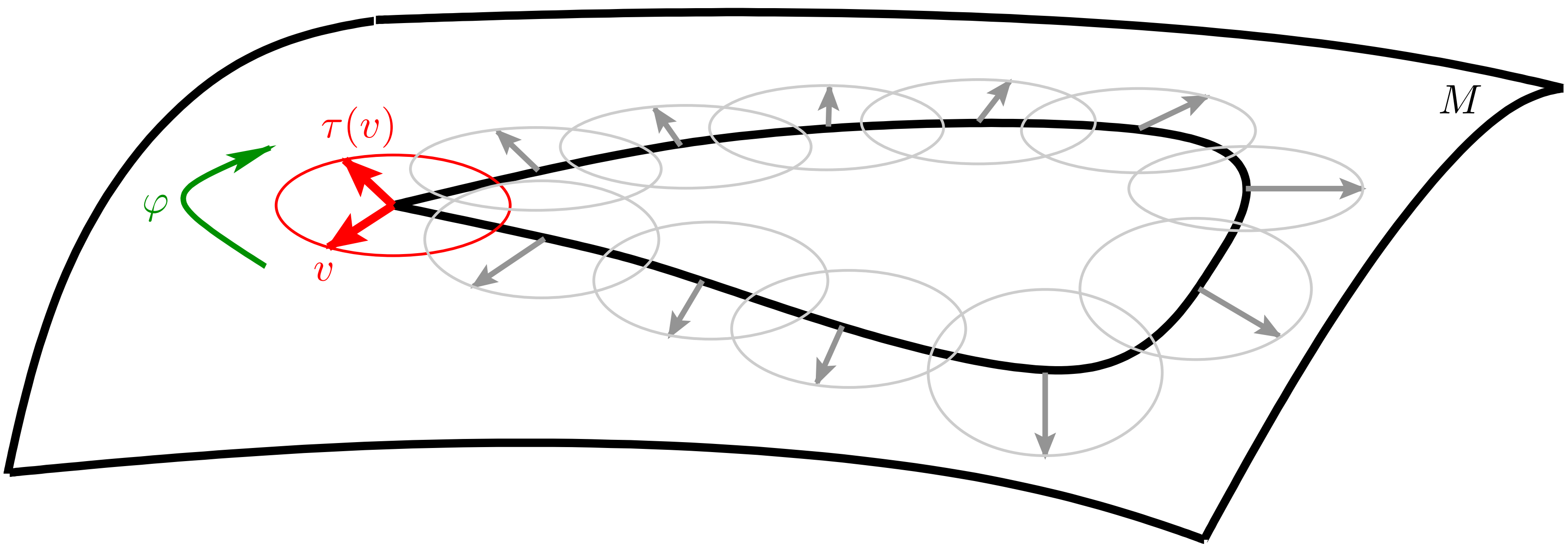

Riemann manifold: $\varphi \in O(n)$

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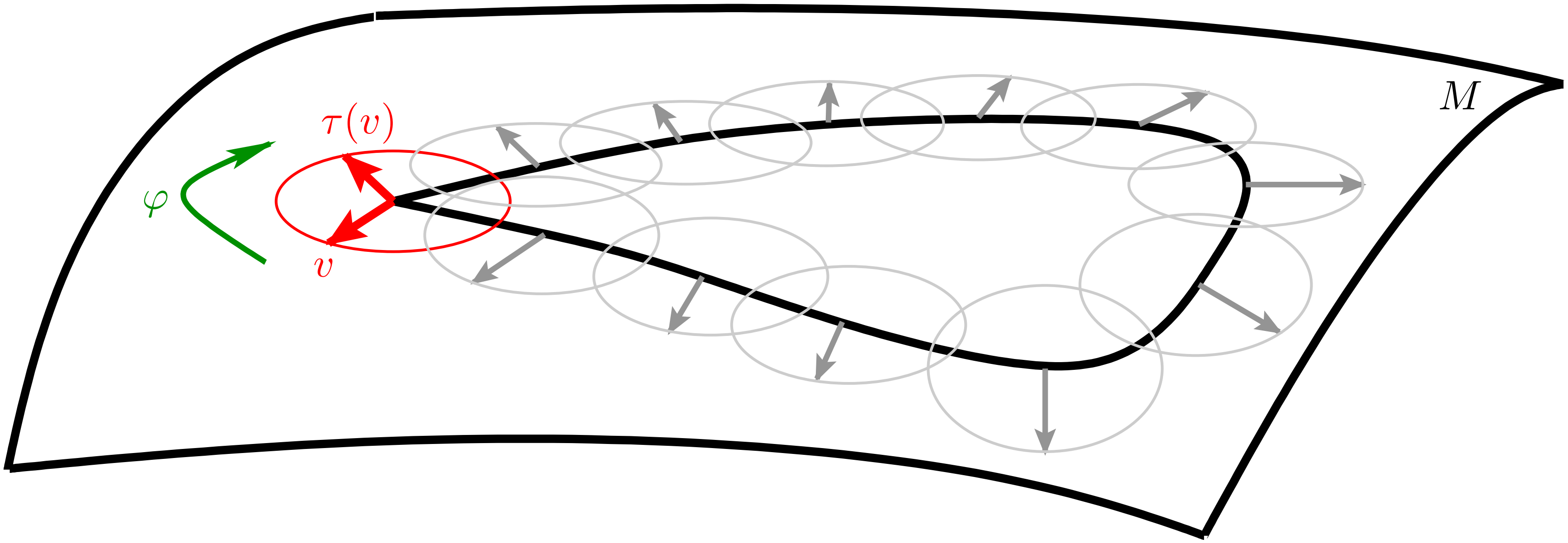

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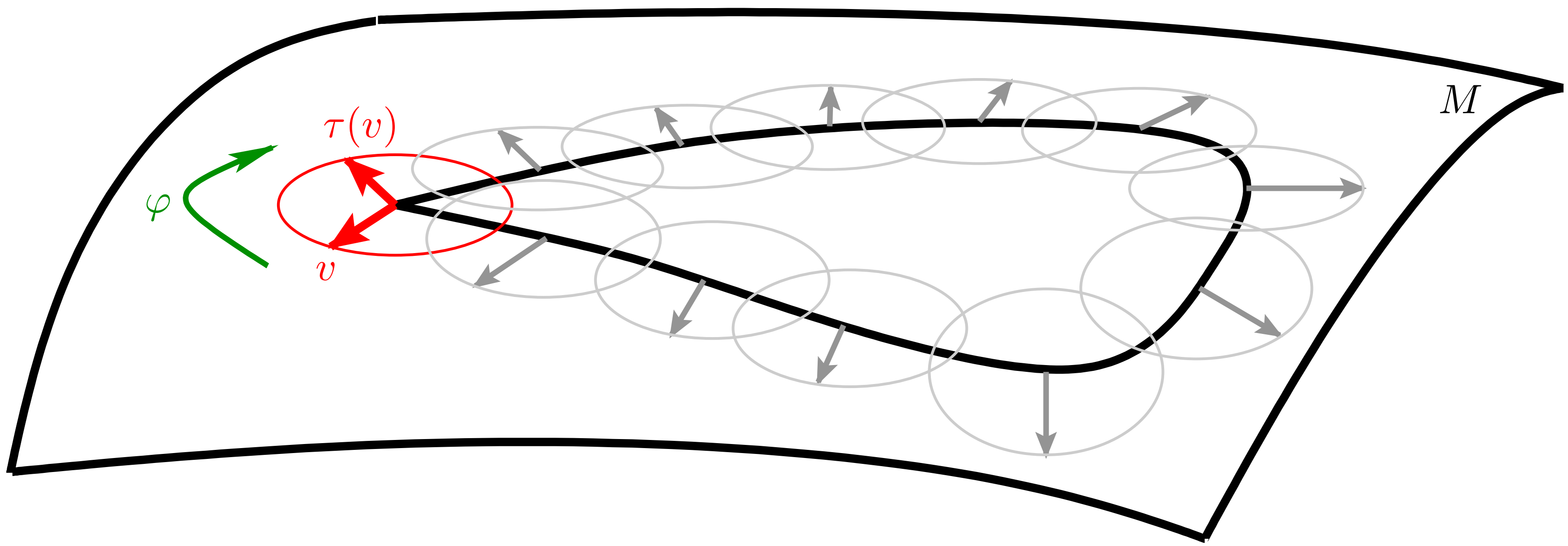

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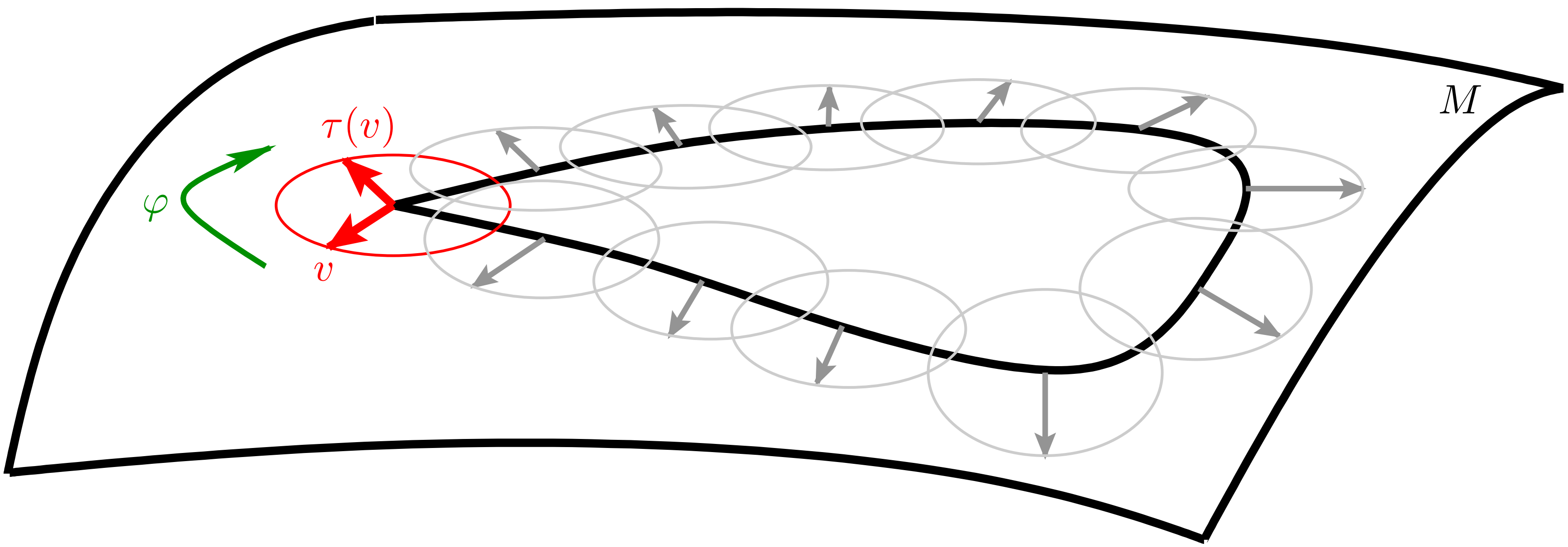

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$$\Big\langle \exp(\mathfrak{X}(\mathbb{S}^1))\Big\rangle \subset \overline{\mathsf{Hol}_0(M)} \subset \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$$

- $\Big\langle \exp(\mathfrak{X}(\mathbb{S}^1))\Big\rangle$ normal subgroup in the simple group $\mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1)$

Theorem. The holonomy group of a projectively flat, non-Riemannian Finsler 2-manifolds of constant curvature is maximal and

$$\overline{\mathsf{Hol}(M)} \cong \mathsf{Diff}^{\infty}_{+}(\mathbb{S}^1).$$

Proof:

- $F_a(x,y) = \dfrac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2}}{1-|x|^2} \pm \left(\dfrac{\langle x,y\rangle}{1-|x|^2} + \dfrac{\langle a,y\rangle}{1+\langle a,x\rangle}\right)$
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References

[1] Hubicska B., Muzsnay Z.: *The holonomy groups of projectively flat Randers two-manifolds of constant curvature*, submitted, (2017),

[2] Muzsnay Z., Nagy P.: *Finsler manifolds with non-Riemannian holonomy*, Houston Journal of Mathematics, (2012),

[3] Muzsnay Z., Nagy P.: *Projectively flat Finsler manifolds with infinite dimensional holonomy*, Forum Mathematicum, (2015),

[4] Muzsnay Z., Nagy P.: *Characterization of projective Finsler manifolds of constant curvature having infinite dimensional holonomy group*, Publ. Math. Debrecen, (2014),

[5] Muzsnay Z., Nagy P.: *Finsler 2-manifolds with maximal holonomy group of infinite dimension*, Differential Geometry and its Applications, (2015),

[6] Herman, M.R.: *Sur le groupe des difféomorphisme du tore*, Ann. Inst. Fourier 23 (1973), 75–86.

[7] Shen, Z.: *Projectively flat Finsler metrics with constant flag curvature,* Trans. Amer. Math. Soc. 355 (2003), 1713–1728.

[8] Shen, Z.: *Projectively flat Randers metrics with constant flag curvature,* Math. Ann. 325 (2003), 19–30

References

[1] Hubicska B., Muzsnay Z.: *The holonomy groups of projectively flat Randers two-manifolds of constant curvature*, submitted, (2017),

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