Blocking sets for substructures and reducing the diameter

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This talk is based on joint work with AART BLOKHUIS and LEO STORME. It is about a very special substructure, other substructures can also be considered, so there are infinitely many similar problems.

Special thanks are due to TAMÁS HÉGER who drew the figures.

Tamás Héger

Motivation of the problem

Definition

A *blocking set* in an incidence structure is a set of points which intersects each line.

In hypergraph terminology, they are called 1-covers.
First, blocking sets were studied in projective planes.

Proposition

In Π_q any blocking set has at least $q + 1$ points. In case of equality, the blocking set is a line.

In projective planes, a blocking set is called *trivial* if it contains a line.

Some illustrative results

For projective planes we have the following results.

Theorem (Bruen; Pelikán)

For a non-trivial blocking set of Π_q we have $|B| \geq q + \sqrt{q} + 1$, and in case of equality we have a subplane of order $\sqrt{q}$ (Baer subplane).

For the plane $\mathrm{PG}(2, q)$ much better results are known.

Theorem (Blokhuis)

Let B be a non-trivial blocking set of $\mathrm{PG}(2, q)$. If q is a prime, then $|B| \geq 3(p + 1)/2$. If $q = p^h$, is not a prime and h is odd, then $|B| \geq q + \sqrt{pq} + 1$.

Blocking sets of substructures

The question of bl. sets w.r.t. substructures was posed in full generality by MAZZOCCA.
A natural substructure of a projective plane is the affine plane.

Theorem (Jamison; Brouwer-Schrijver)

A blocking set of $\mathrm{AG}(2, q)$ has at least $2q - 1$ points.

So, we can consider a substructure consisting of some points and some lines in a projective plane and ask for results on the smallest size of a blocking set, or constructions for a blocking set.
For Hermitian curves: BLOKHUIS, BROUWER, JUNGNICKEL, KRČADINAC, ROTTEY, STORME, SzT, VANDENDRIESSCHE

More illustrative results, q odd

BOROS, FÜREDI, KAHN considered a conic and wanted to block all the secants and tangents. The smallest bl. set has at least $q+1$ points. The blocking sets of minimum size contain m points from the conic and suitably chosen $q+1-m$ points from a line. (A special case is an external line of a conic.)
AGUGLIA, KORCHMÁROS wanted to block all external lines of a conic: at least $q-1$ points. Points of a chord, and two sporadic examples for $q = 5, 7$.
More variations: AGUGLIA, KORCHMÁROS block external lines and tangents: at least q points are needed (points of a tangent and two other (infinite) classes of examples.
Even further results: AGUGLIA, GIULIETTI, KORCHMÁROS, MONTANUCCI, SICILIANO

Our problem

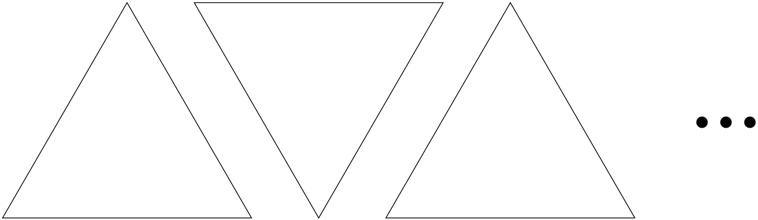
Our question: consider t disjoint Baer subplanes and consider the set of points and lines of these subplanes. In a plane of order q^2 this is a set of $t(q^2 + q + 1)$ points, and we wish to block lines meeting it in $q + t$ points. (Other lines meet it in t points.) We will cal such a set a relative blocking set.

With this set of lines, this is just a non-disjoint union of proj. planes, but two planes do not interact too much. This means that a line of one of the proj. planes only intersects the other ones in 1-1 point.

Our problem from another viewpoint

Let us call the discrete diameter of a hypergraph the size of the largest hyperedges (in our example of disjoint Baer subplanes, it is $q + t$).
We wish to delete as few points as possible, so that the diameter gets smaller. Clearly, the set of deleted points form a blocking set w.r.t. the hyperedges of largest size. We call the set of deleted point a DDR-set.

Our substructure



A stability result for lines in Π_q

We need results about sets with few 0-secants.

Theorem (Erdős and Lovász)

A point set of size q in Π_q, with less than $\sqrt{q+1}(q+1-\sqrt{q+1})$ 0-secants always contains at least $q+1-\sqrt{q+1}$ points from a line.

Stability version of BRUEN's theorem:

Theorem (Erdős and Lovász)

If S is a set of $q+k$ points a projective plane of order q and the number of 0-secants is less than $([\sqrt{q}]+1-k)(q-[\sqrt{q}])$, where $k \leq \sqrt{q}+1$, then the set contains at least $q+k-[\sqrt{q}]+1$ collinear points.

The result is sharp for q square: deleting $\sqrt{q}+1-k$ points from a Baer subplane gives this number of 0-secants.

For planes PG$(2,q)$, with q prime.

Theorem (Weiner-SzT)

Let B be a set of points of $\mathrm{PG}(2,q)$, $q=p$ prime, that has at most $\frac{3}{2}(q+1)-\beta$ ($\beta>0$) points. Suppose that the number of 0-secants, δ is less than $(\frac{2}{3}(\beta+1))^2/2$. Then there is a line that contains at least $q-\frac{2\delta}{q+1}$ points.

Note that for $|B|=cq$, $c\geq 1$ the bound on δ in the above theorem is $c'q^2$.

Trivial examples for our problem

Example 0: a fractional DDR-set. Give weight $1/(q+t)$ to every point, then we get a (fractional) set of size $t(q^2+q+1)/(q+t)$.
Example 1: One can take a line in each Baer subplane. This has $t(q+1)$ points and is clearly a blocking set w.r.t. the long lines.

Example 2: Another trivial example: take one of the Baer subplanes. Has size q^2+q+1.

Which one is better? For small t the first one, for large t (close to q^2-q+1), the second one. Both are close to the fractional covering number.

Definition

Let $f_1(q)$ be the threshold such that Example 1 is best for $t \leq f_1(q)$, $f_2(q)$ is the other threshold s.t. Example 2 is best for $t \geq f_2(q)$.

Clearly, $f_1(q) \leq q$.

A folklore result

A (folklore) results that will be used at several places in our paper.

Theorem

Let B_1 and B_2 be two disjoint Baer-subplanes in $\mathrm{PG}(2, q^2)$, let P be a point in B_1, then the extensions of the Baer-sublines in B_1 through P all intersect B_2 in one point, and these $q + 1$ points determine a conic.

A better upper bound for f_1

We just want to block the $q+t$-secants by t sublines minus one point. The point we wish to delete is on $q+t$ long lines, $t-1$ of them are sublines in another Baer subplane, so they are blocked. Consider the lines through the point: they meet the other Baer subplanes in a CONIC. In the remaining subplanes the sublines can take care of two of these lines (if they meet the conic in two points), so we need $2(t-1) \geq q$.

Therefore, $f_1(q) \leq (q+2)/2$.

A lower bound for f_1

We are going to show that $f_1(q) \geq \sqrt{q}/2$. Actually, we prove also a stability version. Let $B_1, \ldots B_t$ be our Baer subplanes.

Theorem

If B is a relative blocking set, $t \leq \sqrt{q}/2$ and $|B| \leq t(q+1)$ then B contains a Baer-subline in each B_i.

and the stability version

Theorem

If $q > 3437$ and $|B| \leq t(q+1) + \sqrt{q}$ then B contains a Baer-subline in each B_i, or B intersects all but one B_i in a Baer-subline and the last one in a Baer-Baer-subplane.

We shall only sketch the proof of the first Theorem (when $|B| = t(q+1)$).

Definition

A subplane B_i is small if $|B \cap B_i| \le q + \sqrt{q}/2$. Otherwise, it is called large.

Lemma

If $|B \cap B_i| \le q - t$, then $|B| > t(q+1) + \sqrt{q}$.

Follows easily from the number of 0-secants, it implies $|B \cap B_i| \ge q - \sqrt{q}/2$.

Small planes have a large collinear subset

Lemma

There is no subplane B_i with $\geq t(q+1)$ 0-secants.

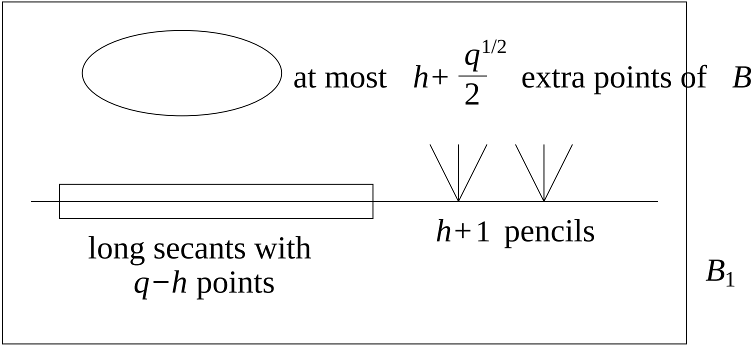
Lemma

A small plane has a collinear subset of size at least $q - t$.

Notation. 1) Let $q - h$ be the size of the smallest collinear set on a long line in a small plane. (So $h \leq t$).
2) Let s denote the number of small planes, w stands for $\sqrt{q}/2$.

Lemma

We have $s \geq \frac{t(w-1)}{w+h}$, and in total there are at most h big planes.


at most $h+\frac{q^{1/2}}{2}$ extra points of B
long secants with $q-h$ points
$h+1$ pencils
B_1

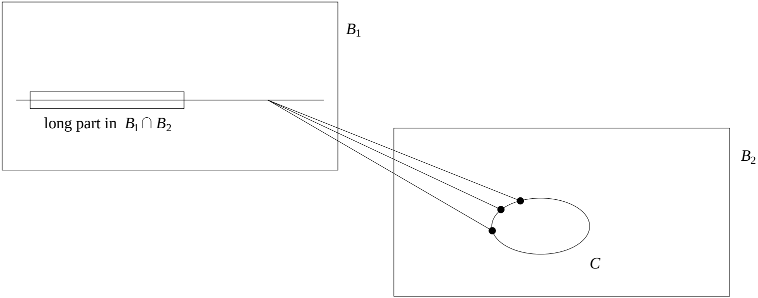
The case $h > 0$

Suppose B_1 is the Baer subplane with the shortest long secant, of size $q - h$.
Step 1: $B \cap B_1$ has at least $(h+1)(q-w-h)$ 0-secants, since we see $h+1$ disjoint pencils on the points of the long secants.
Step 2: As small planes contain long lines, and the lines of a pencil of B_1 intersect ihe other subplanes B_i in a conic, the lines of the pencils mostly have to be blocked in big planes.
This gives (if we do everything more precisely),

$$|B| \geq (q-h) + (s-1)(q-3h-2) + (h+1)(q-w-h).$$

Using the bound on the number of big planes and some computation gives a contradisction in this case.

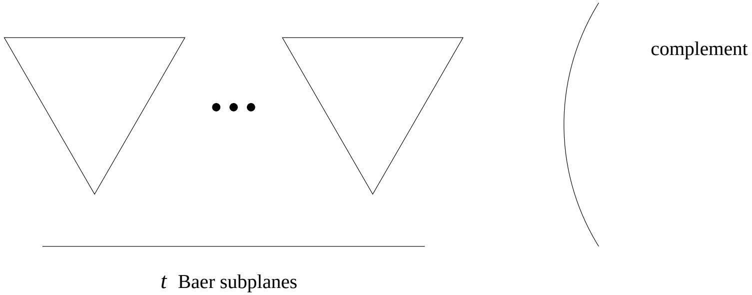
Lines of B_1 have to be blocked in another subplane



The case $h = 0$

In this case all planes are small and have at least q (collinear) points, if there is a plane with just q points, then of the q passants at most $2(t-1)$ are blocked by points on a long secant, and at most t by one of the remaining points, but $q > 3t-2$. If all planes have exactly $q+1$ points, then almost the same argument shows that they form a line in each of the planes.

For t large the complement is considered



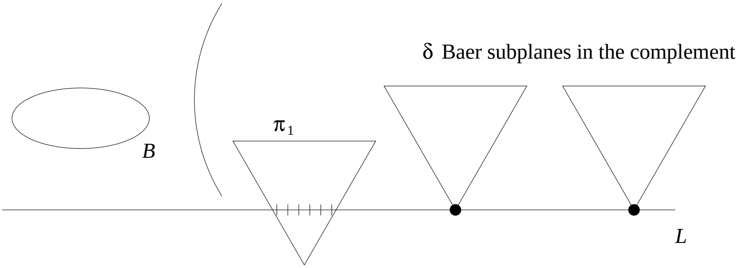
A lower bound on f_2

BLOKHUIS, STORME, SzT: If $t = q^2 - q + 1 - \delta$, then the complement is a τ-fold bl. set, and for $\delta \leq cq^{1/6}$ it is the union of δ Baer subplanes.

Now if our bl. set has $|B| \leq q^2 + q + 1$, then it is either a bl. set in the plane $\mathrm{PG}(2, q^2)$ (and it is a Baer subplane) or it can be completed to an affine blocking set by adding one line in $\delta - 1$ of the missing subplanes and an affine bl. set in the missing subplane of which the missing line (line at infty of the affine plane) belongs to.

Note that the Baer subplane is not necessarily one of the subplanes of the partition.

When the complement is the union of disjoint Baer subplanes



The function f_2: some improvements

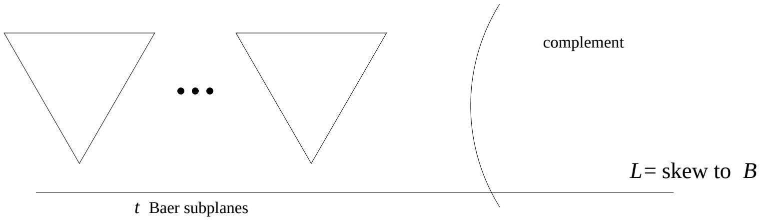
Suppose there are t pairwise disjoint Baer subplanes, $t = q^2 - q + 1 - \delta$, δ small.

Lemma

The complement has intersection sizes δ and $q + \delta$ with the lines of $\mathrm{PG}(2, q^2)$.
Every point R of the complement lies on $q + \delta$ long lines, lines having $q + \delta$ lines of the complement.

What can we say about examples of size less than $q^2 + q + 1$?
They cannot be a blocking set, since they cannot contain a line.
So, let L be a line skew to B.
We will make an affine blocking set for
$\mathbb{A} := \mathrm{AG}(2, q^2) = \mathrm{PG}(2, q^2) \setminus L$.

There is a line disjoint from the complement



Construction of the affine blocking set

The complement has 1) size $\delta(q^2+q+1)$
2) intersection sizes δ or $q+\delta$ with lines
3) every point R in the complement belongs to $q+\delta$ lines containing $q+\delta$ points of the complement.
We only look at the lines containing $q+\delta$ points of the complement since the lines sharing δ points with the complement are blocked by B. We again construct an affine blocking set of $\mathbb{A}$ by using Lovasz' fractional cover bound.
Every point of the complement in $\mathbb{A}$ lies on $q+\delta$ different $(q+\delta)$- or $(q+\delta-1)$-secants.
The fractional cover bound tells us that if we give all points in intersection of the complement with $\mathbb{A}$ weight $1/(q+\delta-1)$, then the fractional covering number is at most

$$\tau^* \leq \frac{\delta(q^2+q+1)-(q+\delta)}{q+\delta-1}.$$

Construction of the affine blocking set II

By the result of Lovász, we have a blocking set of size at most $\tau^* \log(q+\delta)$.
So by adding this to B, we get an affine blocking set of $\mathbb{A}$. So

$$q^2 + q + 1 + \left(\frac{\delta(q^2+q+1) - (q+\delta)}{q+\delta-1} \right) \log(q+\delta) \geq 2q^2 - 1.$$

Here the fraction is at most δq since $\delta \geq 1$, so $\delta \geq \dfrac{q}{\log q}$.

Assume $\delta \leq q$, then $\delta \geq \dfrac{q^2-q-2}{q\log(2q)}$. This means that for such a δ, there can be no line disjoint from our relative blocking set.

A remark for q prime

When q is a prime, we have a better stability result (Weiner, SzT), so we can reach $f_1 \geq cq$. More precisely,

Theorem

There are positive constants $c < d$ so that for $q = p > q_0$ and $t \leq cp$, $|B| \leq cp(p+1) + dp$, the relative blocking set B contains a Baer-subline in each Baer-subplane B_i.

Unfortunately, we could not prove a sharp result here (so we could not reach $d = 1/2$).
However, this remark shows that $f_1(p) \geq cp$, so the order of magnitude is correct.

Thank you for your attention!