The crossing lemma for multigraphs

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joint work with János Pach

Graph G Crossing number $cr(G)$: min. number of edge crossings over all drawings of G (on the plane)

<u>Crossing Lemma</u> (Ajtai-Chvátal-Newborn-Szemerédi 82, Leighton 83)

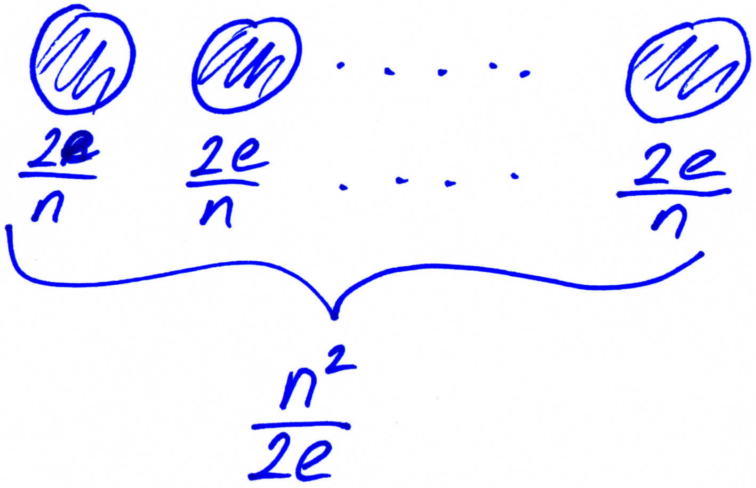
G simple graph, n vertices, $e \geq 4n$ edges $\Longrightarrow$

$$cr(G) \geq \frac{1}{64} \cdot \frac{e^3}{n^2}$$

$$e \geq 4n \Rightarrow cr(G) \geq \frac{1}{64}\cdot\frac{e^3}{n^2}$$

Asymptotically tight:

G:



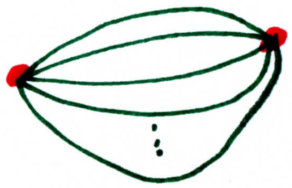
$$n(G) = n$$

$$e(G) \approx \binom{\frac{2e}{n}}{2}\cdot\frac{n^2}{2e} \approx e$$

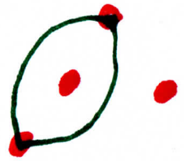
$$cr(G) \leq \binom{\binom{\frac{2e}{n}}{2}}{2}\cdot\frac{n^2}{2e} \approx \frac{e^3}{n^2}$$

Does the Crossing Lemma hold for multigraphs?

NO:

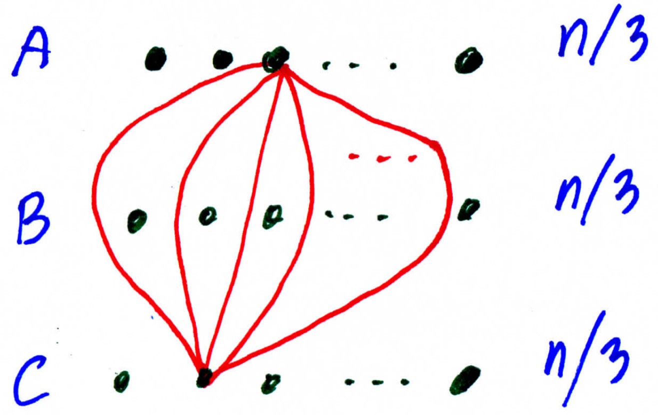


Suppose that for any two parallel edges:



Both regions contain a vertex.

<u>Still does not hold</u>:



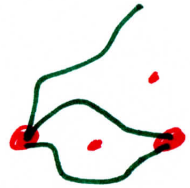
All edges between A and C, each in $n/3$ versions. $n(G)=n$

$$e(G)=\left(\frac{n}{3}\right)^3 \qquad \frac{e(G)^3}{n(G)^2}\approx n^7$$

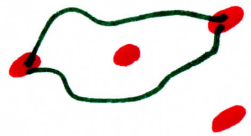
$$cr(G)\leq e(G)^2 \approx n^6$$

Simple drawing of multigraph G:

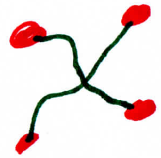
1. parallel edges, edges with common endpoint, do not cross:



2. for any two parallel edges, both regions they define contain a vertex:



3. independent edges cross at most once:



Simple crossing number $scr(G)$ of multigraph G:

minimum number of crossings over all simple drawings.

Theorem (Pach T., conjectured by M. Kaufmann)

G multigraph, $e(G) \geq 4n(G) \Longrightarrow$

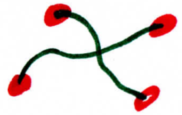
$$scr(G) \geq c \cdot \frac{e^3}{n^2}$$

(asympt. tight)

Crossing Lemma (simple graph, $e \geq 4n \Rightarrow cr(G) \geq \frac{1}{64} \frac{e^3}{n^2}$)

nicest proof:

1. $cr(G) = 0 \Rightarrow G$ planar $\Rightarrow e \leq 3n-6$
2. $cr(G) \geq e - 3n$

Proof: induction on e. Best drawing: $e > 3n \Rightarrow$ there is a crossing 

remove edge in crossing

$cr(G') \leq cr(G) - 1$ $\quad e(G') = e(G) - 1$ .... induction.

3. <u>Random sampling</u>: $e \geq 4n$, best drawing. Let $p = \frac{4n}{e} \leq 1$.

Take each vertex independently with prob. p, G': induced subgraph.

$E(n(G')) = pn \quad E(e(G')) = p^2 \cdot e \quad E(cr(G')) \leq p^4 \cdot cr(G)$

$cr(G') \geq e(G') - 3n(G')$

$p^4 cr(G) \geq p^2 \cdot e - 3p \cdot n \qquad p = \frac{4n}{e}$

$cr(G) \geq \frac{1}{64} \cdot \frac{e^3}{n^2}$

Theorem (PT): G multigraph, $e \geq 4n \implies \mathrm{scr}(G) \geq c \cdot \frac{e^3}{n^2}$

Try previous proof.

1. $\mathrm{scr}(G) = 0 \implies e \leq 3n-6$ TRUE!

   Isolated vertices: induction. (or add extra edge)

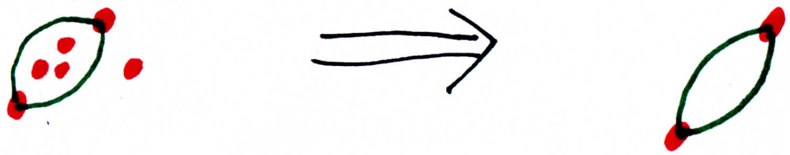
   No isolated vert.: no  each face: ≥ 3 sides.

   Euler formula + standard calculation.

2. $\mathrm{scr}(G) \geq e - 3n$ TRUE!

   (remove edge: remains simple drawing)

3. Random sampling: NO! Proof collapses, could not fix it.



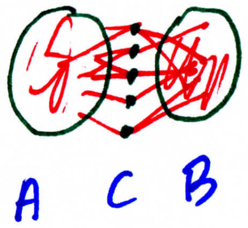
simple non-simple

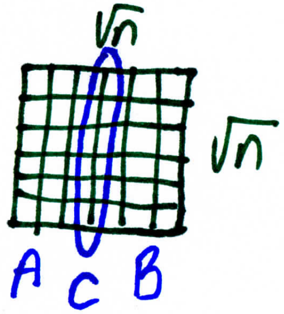
Different approach:

Lipton-Tarjan separator theorem:

G planar, n vertices: $\Longrightarrow$ there is partition $A \cup B \cup C = V(G)$:

$|A|, |B| \le \frac{2}{3} n$, $|C| \le 4 \cdot \sqrt{n}$, no edge between A and B.



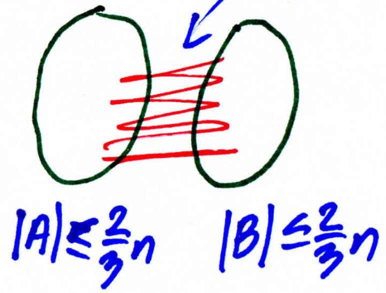


C: vertex-separator

bisection width: $b(G) = \min_{\substack{V = A \cup B \\ A, B \le 2/3 n}} |E(A,B)|$

minimum $= b(G)$

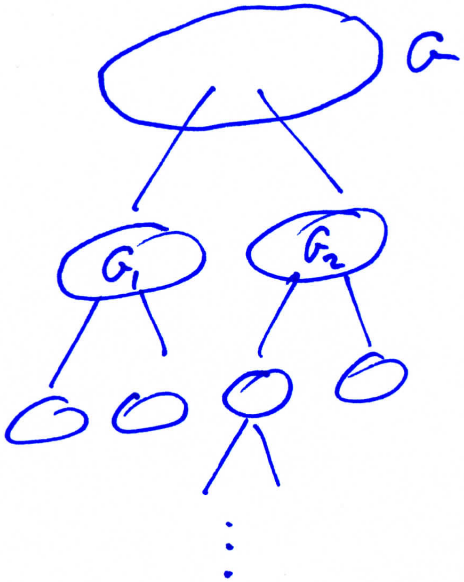
(edge-separator)



Theorem (Pach-Shahrokhi-Szegedy, Alon-Seymour-Thomas, Gazit-Miller...)
G: n vertices, degrees $d_1, d_2 \dots d_n$

$$\Longrightarrow \quad b(G) \le 10 \cdot \sqrt{cr(G)} + 2 \cdot \sqrt{\sum d_i^2}$$

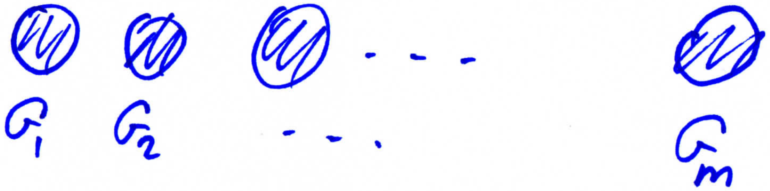
Crossing lemma, complicated proof: Want: $cr(G) \geq c \cdot \frac{e^3}{n^2}$



G_i : cut (remove $\sim \sqrt{cr(G_i)}$ edges)
if $cr(G_i)$ is small $\left(o\left(e(G_i)^2\right)\right)$
otherwise STOP

recursively.

Finally:



removed $o(e)$ edges, typical component: $\approx \frac{e}{n}$ vertices, $\approx \frac{e^2}{n^2}$ edges

$$cr(G_i) \approx \frac{e^4}{n^4} \implies cr(G) \geq c \cdot \frac{n^2}{e} \cdot \frac{e^4}{n^4} = c \cdot \frac{e^3}{n^2}$$

Modification: works for $scr(G)$ as well!

Things to check:

1. G multigraph, simple drawing: $e \le n^2$

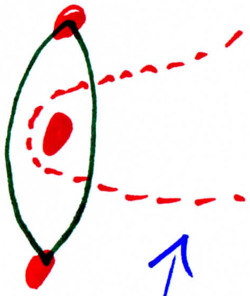
   Proof:

$v \Rightarrow$ cyclic sequence of edges (neighbors):

no $a \dots b \dots a \dots b \Rightarrow d(v) \le 2n-3$

2. After cutting: components remain simple:

   cutting: along actual curve!



can not remove vertex from inside!