The Kneser–Poulsen Conjecture for Uniform Contractions

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Némafilm ...

Notation, terminology

$\mathbf{p} = (p_1, \ldots, p_N) \in \mathbb{E}^{d\times N}$: a configuration (ie. a set, or a sequence) of N points in Euclidean d-space $\mathbb{E}^d$.

$\mathbf{q} \in \mathbb{E}^{d\times N}$ is a contraction of $\mathbf{p} \in \mathbb{E}^{d\times N}$, if $|q_i - q_j| \leq |p_i - p_j|$ for all $1 \leq i < j \leq N$.

$\mathbf{B}[\mathbf{p}] = \bigcap_{i\in N} \mathbf{B}[p_i, 1]$.

Kneser–Poulsen Conjecture ~'54

If $\mathbf{q} = (q_1, \ldots, q_N)$ is a contraction of $\mathbf{p} = (p_1, \ldots, p_N)$ in $\mathbb{E}^d$, then

$$\mathrm{V}_d\left(\mathbf{B}[\mathbf{p}]\right) \leq \mathrm{V}_d\left(\mathbf{B}[\mathbf{q}]\right). \qquad \text{(KP)}$$

Alexander's Conjecture '85

If $\mathbf{q}$ is a contraction of $\mathbf{p}$ in $\mathbb{E}^2$, then

$$\mathrm{perim}\left(\mathbf{B}[\mathbf{p}]\right) \leq \mathrm{perim}\left(\mathbf{B}[\mathbf{q}]\right). \qquad \text{(A)}$$

Habicht and Kneser: for unions, the reversal (A) is FALSE.

Uniform contraction

$\mathbf{q} \in \mathbb{E}^{d\times N}$ is a uniform contraction of $\mathbf{p} \in \mathbb{E}^{d\times N}$ with separating value λ, if

$$|q_i - q_j| \leq \lambda \leq |p_i - p_j| \text{ for all } 1 \leq i < j \leq N. \tag{UC}$$

Motivation

Peter Pivovarov's idea to disprove (KP): sample $\mathbf{p}$ and $\mathbf{q}$ randomly. Show that with $\neq 0$ probability, (KP) is false, while (UC) holds.
[Paouris-Pivovarov, *Random ball-polyhedra ...*, Monatshefte, 2016].

Main result

$k \in [d]$. Let $\mathbf{q} \in \mathbb{E}^{d\times N}$ be a uniform contraction of $\mathbf{p} \in \mathbb{E}^{d\times N}$ with any separating value $\lambda \in (0, 2]$. If $N \geq \left(1+\sqrt{2}\right)^d$ then

$$\mathrm{V}_k\left(\mathbf{B}[\mathbf{p}]\right) \leq \mathrm{V}_k\left(\mathbf{B}[\mathbf{q}]\right). \tag{1}$$

A bit stronger:

Theorem

$k \in [d]$. Let $\mathbf{q} \in \mathbb{E}^{d\times N}$ be a uniform contraction of $\mathbf{p} \in \mathbb{E}^{d\times N}$ with any separating value $\lambda \in (0,2]$. If

(a) $N \geq \left(1+\frac{2}{\lambda}\right)^d$,

or

(b) $\lambda \leq \sqrt{2}$ *and* $N \geq \left(1+\sqrt{\frac{2d}{d+1}}\right)^d$,

then (1) *holds.*

Unions

Theorem

Let $\mathbf{q} \in \mathbb{E}^{d \times N}$ *be a uniform contraction of* $\mathbf{p} \in \mathbb{E}^{d \times N}$ *with some separating value* $\lambda \in (0, 2]$*. If* $N \geq \left(1 + 2d^3\right)^d$ *then*

$$\mathrm{V}_d\left(\bigcup_{i=1}^{N} \mathbf{B}[p_i]\right) \geq \mathrm{V}_d\left(\bigcup_{i=1}^{N} \mathbf{B}[q_i]\right).$$

Proof

Heavy lifting done by Rogers and Bezdek–Lángi on soft ball packings.

Proof of the Main Result – Trivial estimates

$$f_k(d, N, \lambda) := \min\left\{\mathrm{V}_k\left(\mathbf{B}[\mathbf{q}]\right) \ : \ \mathbf{q} \in \mathbb{E}^{d\times N}, |q_i - q_j| \leq \lambda \ \ \forall i, j \in [N], i \neq j\right\},$$

$$g_k(d, N, \lambda) := \max\left\{\mathrm{V}_k\left(\mathbf{B}[\mathbf{p}]\right) \ : \ \mathbf{p} \in \mathbb{E}^{d\times N}, |p_i - p_j| \geq \lambda \ \forall i, j \in [N], i \neq j\right\}.$$

Goal: $f_k \geq g_k$.

Jung's Bound on f_k

Let $d, N \in \mathbb{Z}^+$, $k \in [d]$ and $\lambda \in (0, \sqrt{2}]$. Then

$$f_k(d, N, \lambda) \geq \left(1 - \sqrt{\frac{2d}{d+1}}\frac{\lambda}{2}\right)^k \mathrm{V}_k\left(\mathbf{B}[o]\right).$$

Proof: $\mathbf{q}$ is contained in a ball of radius $\sqrt{\frac{2d}{d+1}}\frac{\lambda}{2}$. Thus, $\mathbf{B}[\mathbf{q}]$ contains a ball of radius ... □

Proof of the Main Result – Trivial estimates

$$g_k(d, N, \lambda) := \max \left\{ \mathrm{V}_k(\mathbf{B}[\mathbf{p}]) \ : \ \mathbf{p} \in \mathbb{E}^{d \times N}, |p_i - p_j| \geq \lambda \ \forall i, j \in [N], i \neq j \right\}.$$

Packing bound on g_k

Let $d, N \in \mathbb{Z}^+$, $k \in [d]$ and $\lambda > 0$.

$$\text{If } N \left(\frac{\lambda}{2}\right)^d \geq \left(1 + \frac{\lambda}{2}\right)^d, \text{ then } g_k(d, N, \lambda) = 0.$$

Proof: $\{\mathbf{B}[p_i, \lambda/2]\}$ is a packing. Thus, taking volume yields that the circumradius of the set $\{p_i\}$ is at least one. Hence, $\mathbf{B}[\mathbf{p}]$ is a singleton or empty. □

An additive Blaschke–Santalo inequality

$X \subset \mathbb{E}^d, \mathrm{cr}(X) \leq \rho$. The ρ-spindle convex hull of X is $\mathrm{conv}_\rho(X) := \mathbf{B}[\mathbf{B}[X, \rho], \rho]$. Easily, $\mathbf{B}[X, \rho] = \mathbf{B}[\mathrm{conv}_\rho(X), \rho]$.

Fodor, Kurusa, Vígh: A Blaschke–Santalo-type inequality for the volume of spindle convex sets, [FKV, *Inequalities* ..., Adv. Geom, '16].

A variation: an additive Blaschke–Santalo-type inequality for spindle-convex sets for intrinsic volumes.

Additive Blaschke–Santalo inequality

$Y \subset \mathbb{E}^d$ a ρ-spindle convex set, $k \in [d]$. Then

$$\mathrm{V}_k\left(Y\right)^{1/k} + \mathrm{V}_k\left(\mathbf{B}[Y, \rho]\right)^{1/k} \leq \rho \mathrm{V}_k\left(\mathbf{B}[o]\right)^{1/k}. \qquad \text{(Bl.}\neq\text{Sa.)}$$

An additive Blaschke–Santalo inequality

$Y \subset \mathbb{E}^d$ a ρ-spindle convex set, $k \in [d]$. Then

$$\mathrm{V}_k(Y)^{1/k} + \mathrm{V}_k(\mathbf{B}[Y,\rho])^{1/k} \leq \rho \mathrm{V}_k(\mathbf{B}[o])^{1/k}.$$

Proof:

Proposition [folklore (?)]

$Y \subset \mathbb{E}^d$ a ρ-spindle convex set. Then

$$Y - \mathbf{B}[Y,\rho] = \mathbf{B}[o,\rho].$$

Combine with the Brunn–Minkowski theorem for intrinsic volumes. □

Proof of the Proposition

$Y \subset \mathbb{E}^d$ a ρ-spindle convex set. Then

$$Y - \mathbf{B}[Y, \rho] = \mathbf{B}[o, \rho].$$

Y spindle-convex, thus, Y slides freely in $\mathbf{B}[o, \rho]$.
Thus, Y is a summand of $\mathbf{B}[o, \rho]$ and so,

$$Y + (\mathbf{B}[o, \rho] \sim Y) = \mathbf{B}[o, \rho],$$

where $\sim$ is the Minkowski difference: $\mathbf{B}[o, \rho] \sim Y := \cap_{y \in Y}(\mathbf{B}[o, \rho] - y)$.
On the other hand, $\cap_{y \in Y}(\mathbf{B}[o, \rho] - y) = -\mathbf{B}[Y, \rho]$. $\square$

A non-trivial bound on g

$$g_k(d, N, \lambda) := \max\left\{ \mathrm{V}_k\left(\mathbf{B}[\mathbf{p}]\right) \ : \ \mathbf{p} \in \mathbb{E}^{d\times N}, |p_i - p_j| \geq \lambda \ \forall i, j \in [N], i \neq j \right\}.$$

$$g_k(d, N, \lambda) \leq \max\left\{ 0, \left(1 - \left(N^{1/d} - 1\right)\frac{\lambda}{2}\right)^k \mathrm{V}_k\left(\mathbf{B}[o]\right)\right\}.$$

Proof

A simple fact: $\mathbf{B}[\mathbf{p}] \subseteq \mathbf{B}\left[\bigcup_{i=1}^N \mathbf{B}[p_i, \mu], 1+\mu\right]$.

$$\mathrm{V}_k\left(\mathbf{B}[\mathbf{p}]\right) \leq \mathrm{V}_k\left(\mathbf{B}\left[\bigcup_{i=1}^N \mathbf{B}\left[p_i, \frac{\lambda}{2}\right], 1 + \frac{\lambda}{2}\right]\right) =$$

$$\mathrm{V}_k\left(\mathbf{B}\left[\mathrm{conv}_{1+\lambda/2}\left(\bigcup_{i=1}^N \mathbf{B}\left[p_i, \frac{\lambda}{2}\right]\right), 1 + \frac{\lambda}{2}\right]\right) \leq \text{ by (Bl.}\neq\text{Sa.)}$$

$$\left[\left(1 + \frac{\lambda}{2}\right)\mathrm{V}_k\left(\mathbf{B}[o]\right)^{1/k} - \mathrm{V}_k\left(\mathrm{conv}_{1+\lambda/2}\left(\bigcup_{i=1}^N \mathbf{B}\left[p_i, \frac{\lambda}{2}\right]\right)\right)^{1/k}\right]^k \leq$$

Goal: $g_k(d, N, \lambda) \le \left(1 - \left(N^{1/d} - 1\right) \frac{\lambda}{2}\right)^k \mathrm{V}_k\left(\mathbf{B}[o]\right).$

$$\mathrm{V}_k\left(\mathbf{B}[\mathbf{p}]\right) \le \ldots \le$$

$$\left[\left(1+\frac{\lambda}{2}\right)\mathrm{V}_k\left(\mathbf{B}[o]\right)^{1/k} - \mathrm{V}_k\left(\mathrm{conv}_{1+\lambda/2}\left(\bigsqcup_{i=1}^{N}\mathbf{B}\left[p_i, \frac{\lambda}{2}\right]\right)\right)^{1/k}\right]^k \le$$

$$\left[\left(1+\frac{\lambda}{2}\right)\mathrm{V}_k\left(\mathbf{B}[o]\right)^{1/k} - \frac{\lambda}{2}N^{1/d}\mathrm{V}_k\left(\mathbf{B}[o]\right)^{1/k}\right]^k.$$

In the last step, we used:

$$\mathrm{V}_d\left(\mathrm{conv}_{1+\lambda/2}\left(\bigsqcup_{i=1}^{N}\mathbf{B}\left[p_i, \frac{\lambda}{2}\right]\right)\right) \ge \mathrm{V}_d\left((N^{1/d}\lambda/2)\mathbf{B}[o]\right), \qquad \text{(VOL)}$$

and a general isoperimetric inequality: *among all convex bodies of a given volume, the ball has the smallest V_k.*

Thus, we can replace V_d by V_k in (VOL). □

Completing the proof of the Main Result

Combine the bounds on f_k and g_k. $\square$

Strong contractions

Unconditional body: symmetric about each of the d coordinate hyperplanes.

Strong contraction: contraction in each coordinate.

Theorem

$K_1, \ldots, K_N$ unconditional convex bodies in $\mathbb{E}^d$. $\mathbf{q} \in \mathbb{E}^{d\times N}$ a strong contraction of $\mathbf{p} \in \mathbb{E}^{d\times N}$. Then

$$\mathrm{V}_d\left(\bigcup_{i=1}^{N}(p_i + K_i)\right) \geq \mathrm{V}_d\left(\bigcup_{i=1}^{N}(q_i + K_i)\right),$$

and

$$\mathrm{V}_d\left(\bigcap_{i=1}^{N}(p_i + K_i)\right) \leq \mathrm{V}_d\left(\bigcap_{i=1}^{N}(q_i + K_i)\right).$$

Picture time!

Do we need unconditionality? Yes.

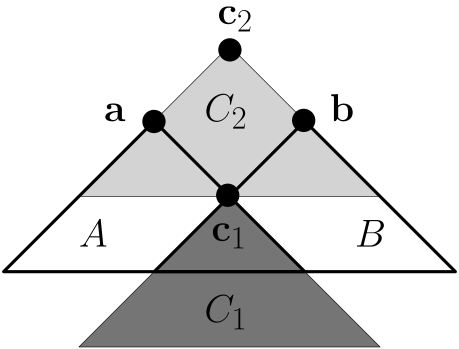

Figure :
1st family: A, B, C_1;
2nd family: A, B, C_2.
Translation vectors: $b = -a$, $c_2 = -c_1$.

Both configurations of the three translation vectors are a strong contraction of the other configuration.

Does it work for the perimeter? No.

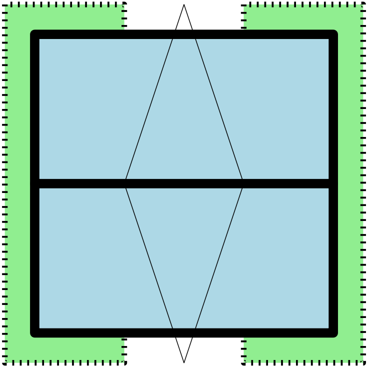

Figure :
1st family: The two green rectangles, the diamond, and the two blue rectangles.
2nd family: The two green rectangles, the diamond, and ONE blue rectangle (counted twice).

Thank you!