Minkowski geometriák izometriáiról

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$$
\begin{tikzcd}
 & \boxed{\boxed{\text{Euclidean space}}} & \\
 & \boxed{\text{inner-product}} \arrow[u, Rightarrow] & \\
\boxed{\text{semi-inner-product}} \arrow[ur] \arrow[d, Rightarrow] & & \boxed{\text{indefinite-inner-product}} \arrow[ul] \arrow[d, Rightarrow] \\
\boxed{\boxed{\text{normed space}}} \arrow[dr] & \text{and} & \boxed{\boxed{\text{pseudo-euclidean space}}} \arrow[dl] \\
 & \boxed{\boxed{\text{Minkowski space}}} &
\end{tikzcd}
$$

The $\left\{ \begin{array}{r} semi-inner-product \\ indefinite-inner-product \end{array} \right.$ on a complex vector space V is a complex function $[x,y] : V \times V \longrightarrow \mathbb{C}$ with the properties $\left\{ \begin{array}{r} s1, s2, s3, s4 \\ i1, i2, i3, i4 \end{array} \right.$

s1=i1: $[x+y,z] = [x,z] + [y,z]$ (additivity of the first argument)

s2=i2: $[\lambda x, y] = \lambda[x,y]$ for every $\lambda \in \mathbb{C}$ (homogeneity of the first argument)

s3: $[x,x] > 0$ when $x \neq 0$ (positivity)

i3: $[x,y] = \overline{[y,x]}$ for every $x, y \in V$ (antisymmetry)

s4: $|[x,y]|^2 \leq [x,x][y,y]$ (Cauchy-Schwartz inequality)

i4: $[x,y] = 0$ for every $y \in V$ then $x = 0$. (nondegeneracy)

Euklideszi eset jellemzése

Félskaláris szorzattal: Euklideszi esetben a félskaláris szorzat skaláris szorzat

- Az első argumentumban is additív.

Ortogonalitásokkal: két különböző jellegű ortogonalitás megegyezik

- Vannak homogén és szimmetrikus ortogonalitások,

  (Busemann, Birkhoff, James illetve isosceles, Pitagoraszi, Radon sík)

Biszektorral, egységgömbbel: Biszektor hipersík, egységgömb ellipszoid

Ellipszoid karakterizációs tételek:

- K ellipszoid ha minden hipersíkkal való metszete ellipszoid
- K ellipszoid, ha határának minden eltoltjával való metszete hipersíkot feszít ki.

Izometria csoporttal jellemzés

H. Auerbach, Sur une propiérté caractéristique de l'ellipsoide, Studia Math. 9 (1940) 17–22.

Corollary 5 *([27,2]). If the isometry group of a Minkowski space is transitive on the unit ball of the space then the unit ball is an ellipsoid and the space is Euclidean.*

Theorem 11 *([7,27,21]). If $(V, \|\cdot\|)$ is a Minkowski plane that is non-Euclidean, then the group $\mathcal{I}(2)$ of isometries of $(V, \|\cdot\|)$ is isomorphic to the semi-direct product of the translation group $\mathcal{T}(2)$ of $\mathbb{R}^2$ with a finite group of even order that is either a cyclic group of rotations or a dihedral group.*

J.-L. Garcia-Roig, On the group of isometries and a system of functional equations of a real normed plane, in: Inner Product Spaces and Applications, in: Pitman Research Notes in Mathematics Series, vol. 376, Longman, Harlow, 1997, pp. 42–53.

A.C. Thompson, Minkowski Geometry, Encyclopedia of Mathematics and Its Applications, vol. 63, Cambridge University Press, Cambridge, 1996.

H. Martini, M. Spirova, K. Strambach, Geometric algebra of strictly convex Minkowski planes, Aequationes Math. 88 (1) (2014) 49–66, http://dx.doi.org/10.1007/s00010-013-0204-z.

Megjegyzések

- Minkowski geometria izometriái affin transzformációi a beágyazó Euklidészi térnek
- Minden dimenzióban, minden Minkowski geometria esetén az izometria csoport tartalmazza az eltolások szemi direkt szorzatát az origóra tükrözéssel ($\mathcal{I}$(n)$\supseteq\mathcal{T}$(n)$\rtimes$C(2) , mert $\mathcal{T}$(n) normálosztó)
- Az egyenes körhenger mint egységgömb példát ad olyan nem-Euklideszi Minkowski geometriára, melynek izometria csoportja tartalmazza $\mathcal{T}$(n)$\rtimes$O(n-1)-t. (Azaz a szemi direkt szorzatnak az egységgömböt invariánsan tartó komponense nem véges csoport.)

Totálisan nem-Euklidészi Minkowski tér

Definition 2. A Minkowski n-space is *totally non-Euclidean* if it has no 2-dimensional Euclidean subspace.

Corollary 1. *In a totally non-Euclidean Minkowski n-space every diagonalizable adjoint abelian operator is a scalar multiple of an isometry.*

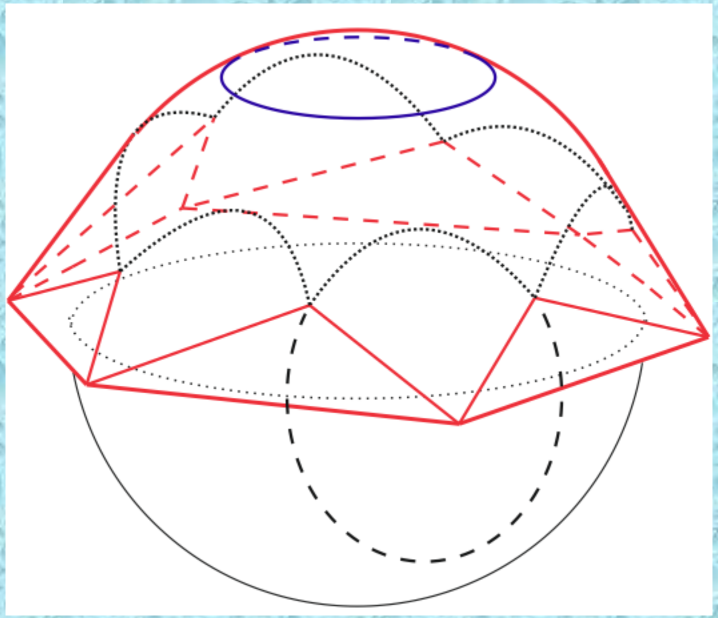
Zs. Lángi, On diagonalizable operators in Minkowski spaces with the Lipschitz property, Linear Algebra Appl. 433/11–12 (2010) 2161–2167.

Nyitott probléma az izometria csoportról

Problem 1. Is it true or not that if for $n \geq 3$ the Minkowski n-space is totally non-Euclidean one (see Definition 4) then its isometry group $\mathcal{I}(n)$ is a semi-direct product of the translation group $\mathcal{T}(n)$ with a finite subgroup of $SL(n)$?

Amit tudunk

Theorem 12. *If the unit ball C of $(V, \|\cdot\|)$ does not intersect a two-plane in an ellipse, then the group $\mathcal{I}(3)$ of isometries of $(V, \|\cdot\|)$ is isomorphic to the semi-direct product of the translation group $\mathcal{T}(3)$ of $\mathbb{R}^3$ with a finite subgroup of the group of linear transformations with determinant ± 1.*



Struktúra tétel

Theorem 10. *Let V be a finite dimensional real Banach space, $U : V \longrightarrow V$ be an isometry on V, and $[\cdot,\cdot]$ be a semi-inner product preserved by U. Then there is a decomposition of the space of form*

$$V = V_1 \oplus \dots V_s \oplus V_{s+1} \oplus \dots \oplus V_l \oplus V_{l+1} \oplus \dots \oplus V_{l+k},$$

where V_i $1 \leq i \leq l$ are U-invariant mutually orthogonal eigenspaces of dimension 1, if $1 \leq i \leq s$ the corresponding eigenvalue is 1, and for $s \leq i \leq l$ the common eigenvalue is -1; moreover $n - l$ is even and the subspaces $V_{l+1}, \dots, V_{l+k}$ are 2-dimensional U-invariant subspaces such that all of them are orthogonal to the 1-dimensional ones. The restriction of U to a 2-dimensional component is a generalized rotation *with respect to an Auerbach basis $\{a_s, b_s\}$ defined by the matrix*

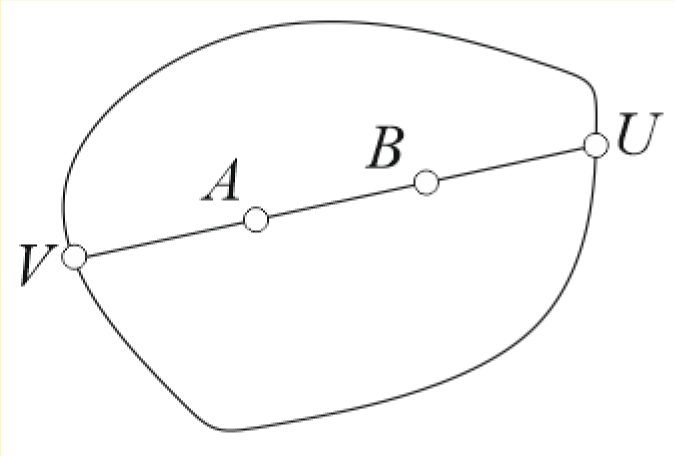
$$[A|_{\mathrm{lin}\{a_s,b_s\}}]_{\{a_s,b_s\}} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \text{ where } 0 < \varphi \leq 2\pi.$$

A Löwner-John ellipszoid szerepe a bizonyításban

- A maximális (minimális) térfogatú K-ba (K-köré) írt ellipszoid egyértelmű, ezért minden izometria invariánsan hagyja a K felszínének a metszetét a John ellipszoid felszínével.
- A probléma affin invariáns így feltehető a John ellipszoidról, hogy az egységgömb, azaz a K-t invariánsan tartó izometriák az O(3) csoport elemei.
- Ha végtelen a K-t invariánsan tartó izometriák részcsoportja, akkor meg tudjuk konstruálni K egy olyan síkmetszetét, melynek minden pontja a metszethez tartozik (azaz a síkmetszet kör).

2 in $\mathcal{I}(3)_0$, so that, G is finite if and only if $\mathcal{I}(3)_0$ is so. Let a point x be a common point of the boundary of C and the boundary of E. (Of course such a point exists.) Let denote by C^+ the closed half sphere containing x and bounded by the hyperplane orthogonal to x through the origin. If the group G is infinite then the orbit of x also contains infinitely many distinct points of form $T_i(x) \in \mathrm{bd}E \cap \mathrm{bd}C^+$ where $T_i \in \mathcal{I}(3)_0$. Since $\mathrm{bd}E \cap \mathrm{bd}C^+$ is compact for every $k \in \mathbb{N}$ there are two indices $i \neq j$ such that $\|T_i(x) - T_j(x)\| \leq 1/k$ implying that $\|T_j^{-1}T_i(x) - x\| \leq 1/k$. Consider the isometry $T_j^{-1}T_i \in SO(3)$. Hence $T_j^{-1}T_i$ is rotation about an axis, say x_k. Thus the points $\left(T_j^{-1}T_i\right)^l(x)$ for $l \in \mathbb{N}$ are on a two dimensional intersection of $\mathrm{bd}E$, so they are also on a circle E_k. This circle through the point x contains a set of points of $\mathrm{bd}C$ with successive distance at most $1/k$ forming an $1/k$-net on it. Let denote by y_k the unit normal vector of the plane of E_k directed by C^+. The set $Y := \{y_k\ k \in \mathbb{N}\}$ is infinite and hence it has a convergent subsequence (y_{k_i}) with limit y. Consider now the circle $E(x,y)$ defined by the intersection of E with the plane through x and orthogonal to y. It has the property that if $z \in E(x,y)$ then for every $\varepsilon > 0$ there is a point u of $\mathrm{bd}E \cap \mathrm{bd}C$ such that $\|z - u\| \leq \varepsilon$. This implies that $E(x,y) \subset \mathrm{bd}E \cap \mathrm{bd}C$ giving a contradiction with our assumption. Thus the group $\mathcal{I}(n)_0$ is finite and the statement is true. $\square$

Hilbert geometriák



Hilbert izometriák

Theorem 4.1. *A finite-dimensional Hilbert geometry on a bounded open convex set is isometric to a normed space if and only if the domain is a simplex.*

Using this correspondence, and the Masur–Ulam theorem, the isometry group of the simplicial Hilbert geometry was determined for $n = 2$ in [9] and for arbitrary n in [17]. Let σ_{n+1} be the group of coordinate permutations on V, let $\rho\colon V \to V$ be the isometry given by $\rho(x) = -x$ for $x \in V$, and identify the group of translations in V with $\mathbb{R}^n$.

Theorem 4.2 ([11], [17]). *If D is an open n-simplex with $n \geq 2$, then*

$$\mathrm{Coll}(D) \cong \mathbb{R}^n \rtimes \sigma_{n+1} \quad \textit{and} \quad \mathrm{Isom}(D) \cong \mathbb{R}^n \rtimes \Gamma_{n+1},$$

where $\Gamma_{n+1} = \sigma_{n+1} \times \langle\rho\rangle$.

A. Karlsson and T. Foertsch, Hilbert metrics and Minkowski norms. *J. Geometry* 83 (2005), no. 1, 22–31.

Theorem 4.3 ([9]). *If the closure of D is strictly convex, then* $\mathrm{Isom}(D) = \mathrm{Coll}(D)$.

P. de la Harpe, On Hilbert's metric for simplices. In *Geometric group theory* (Sussex, 1991), Vol. 1, London Math. Soc. Lecture Note Ser. 181, Cambridge University Press, Cambridge 1993, 97–119.

Theorem 4.4 ([17]). *If* (D, Hil) *is a polyhedral Hilbert geometry, then* $\mathrm{Coll}(D)$ *differs from* $\mathrm{Isom}(D)$ *if and only if D is an open n-simplex, with $n \geq 2$.*

B. Lemmens and C. Walsh, Isometries of polyhedral Hilbert geometries. *J. Topol. Anal.* 3 (2011), no. 2, 213–241.

Theorem 7.3 (Colbois–Verovic). *Let K be a bounded open convex set in $\mathbb{R}^n$ $(n \geq 2)$ whose boundary $\partial\overline{K}$ is a hypersurface of class C^3 which is strictly convex (in the sense that the Hessian is positive definite). Then if $\partial\overline{K}$ is not an ellipsoid, any discrete subgroup of the isometry group $Isom(K, d_k)$ whose action on K is proper is finite.*

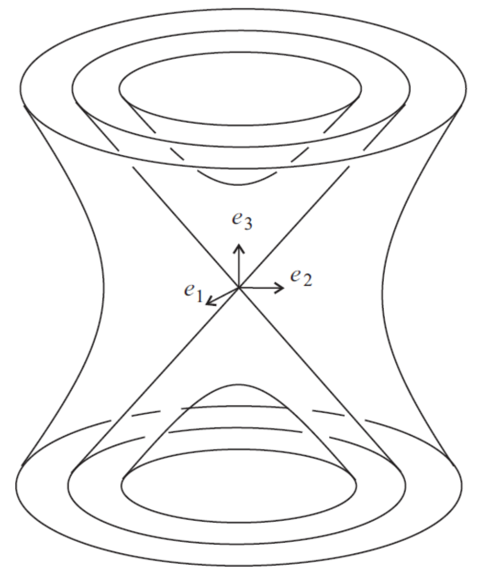
B. Colbois and P. Verovic, Rigidity of Hilbert metrics. *Bull. Austral. Math. Soc.* 65 (2002), 23–34.

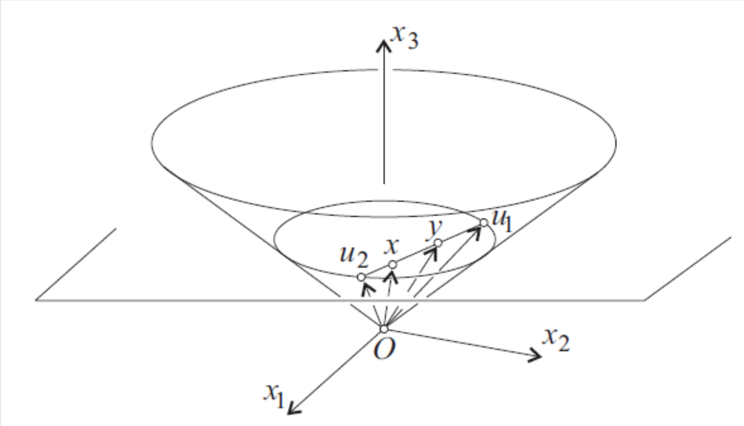
Centrálszimmetrikus Hilbert geometria mint Minkowski geometria feletti hiperbolikus tér

$$\rho_E(x,y)=c(x,y)\,\rho_M(x,y)$$

$$\Downarrow$$

$$\rho_{E,H}(x,y)=\rho_{M,H}(x,y)$$





Kétféle izometria fogalom

Definition 16. A *linear isometry* $f : H^+ \longrightarrow H^+$ of H^+ is the restriction of a linear map $F : V \longrightarrow V$ to H^+ which preserves the Minkowski product and which sends H^+ onto itself.

Definition 18. A *topological isometry* $f : H \longrightarrow H$ of H is a homeomorphism of H which preserves the Minkowski–Finsler distance between each pair of points of H.

Theorem 14. *A linear isometry of H^+ is also a topological isometry on it.*

In the proof of this theorem we also proved that a linear isometry is a Finsler isometry, in the sense that it is a diffeomorphism of H onto H which preserves the Minkowski–Finsler metric function. In a Riemann space the two types of isometries (the topological and the Riemannian one) are equivalent. This is a result of Myers and Steenrod (see in [29]). The analogous theorem on Finsler spaces was proved by Deng and Hou in [30]. In the latter paper it is also stated that the two concepts of isometry are equivalent for a Finsler space.

In the following theorem we impose the condition of linear homogeneity of H^+.

Theorem 15. *Let V be a generalized space–time model. Assume that the space S is strictly convex and smooth and the group of linear isometries of H^+ acts transitively on H^+. Denote the Minkowski–Finsler distance of H^+ by $d(\cdot,\cdot)$. Then the following statement is true:*

$$[a,b]^+ = -ch(d(a,b)) \quad \textit{for } a,b \in H^+.$$

Probléma

Ha kétszeresen tranzitívan hatnak a lineáris izometriák a prehiperbolikus téren (Hilbert geometria egyenlő hosszú szakaszain tranzitívan hatnak), akkor egyszeresen tranzitívan hatnak egy (n-1)-dimenziós metszeten, ami egy gömbje a vízszintes metszet Minkowski térnek, azaz a Minkowski tér Euklideszi, a prehiperbolikus tér hiberbolikus (a Hilbert geometria hiperbolikus geometria).

Egyszeres tranzitív hatásból következik, hogy minden topologikus izometria lineáris, és a metrika függvény szép kapcsolatban van Minkowski-Finsler távolsággal.

Van-e ellipszoidtól különböző konvex test, amire a lineáris izometriák tranzitívan hatnak a prehiperbolikus téren?

Motion of rigid systems in the Euclidean plane (roulettes)

-The plane Σ' moving on the fixed plane Σ

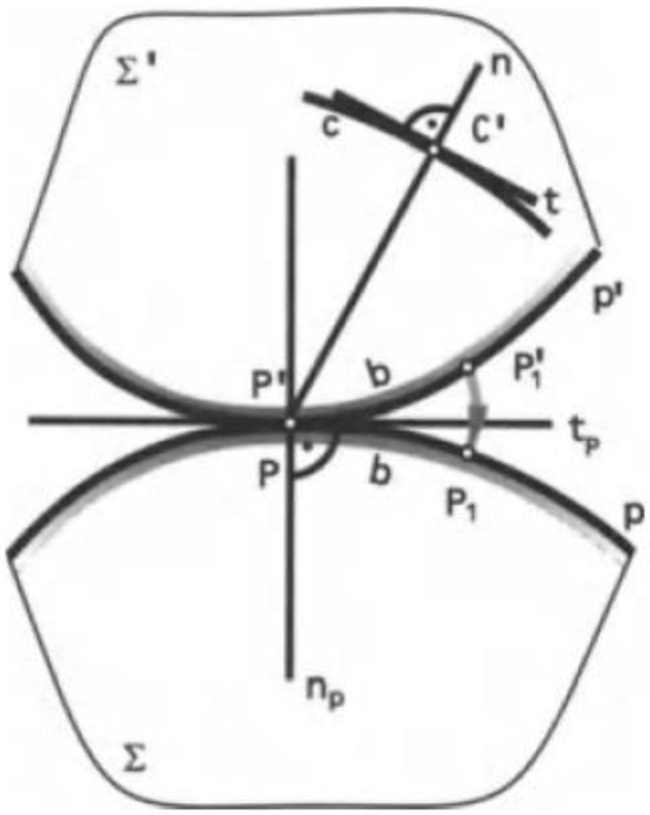
-Fixed frame with coordinates (x,y)

-Moving frame with coordinates (u,v)

-The parameter φ is non-vanishing (the motion is nontranslative)

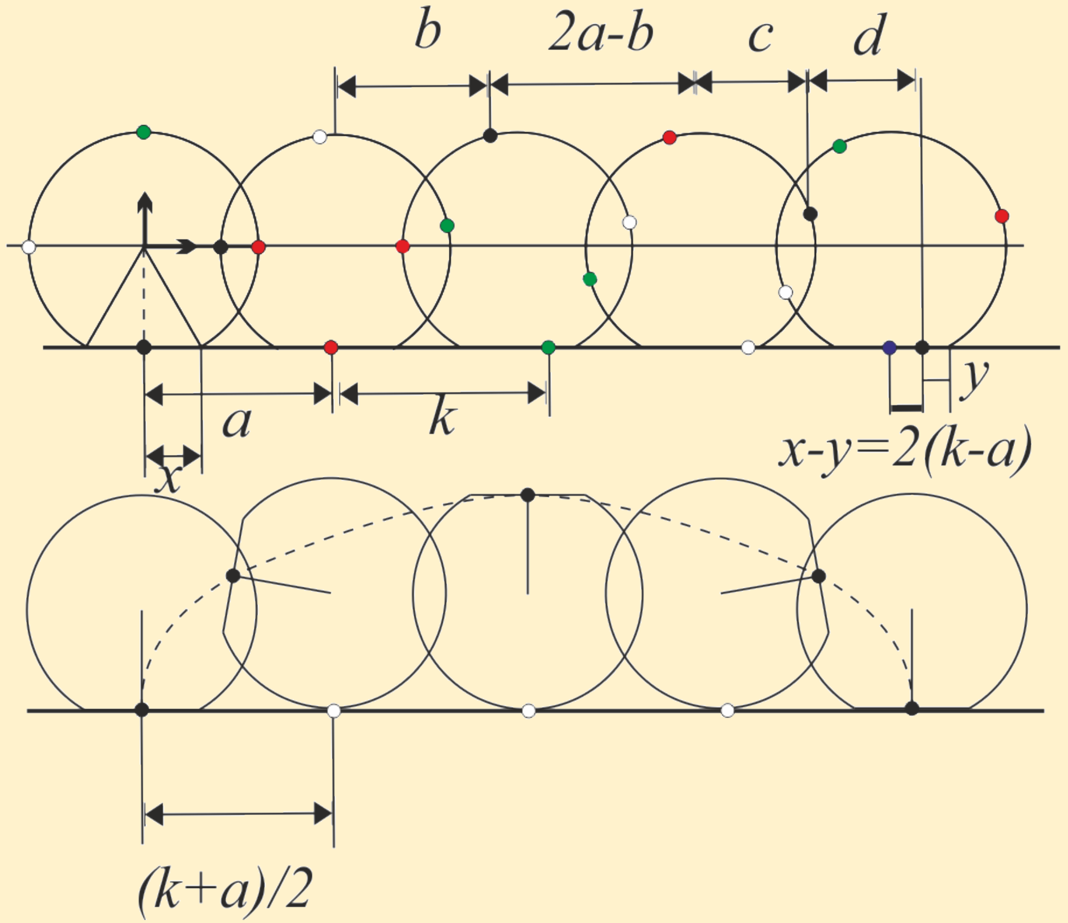
-For every value of φ there is a point of the moving plane for which the velocity vector vanishes. This point (the so-called instantaneous centre) defines in the moving plane the moving polode (centroid) and in the fixed plane the fixed polode (centroid), respectively.

-The velocity vector at a point is orthogonal to the position vector from the instantaneous centre to the point, implying that the move is an instantaneous rotation about the instantaneous centre.



$$x = p + u \cos\varphi - v \sin\varphi\,, \qquad y = q + u \sin\varphi + v \cos\varphi$$

A kerék mozgása I (az ívtávolságok rögzitettek)



$2k=\pi$

$a=k-\arcsin(x)+x$

$b=a-\sin(k-a)$

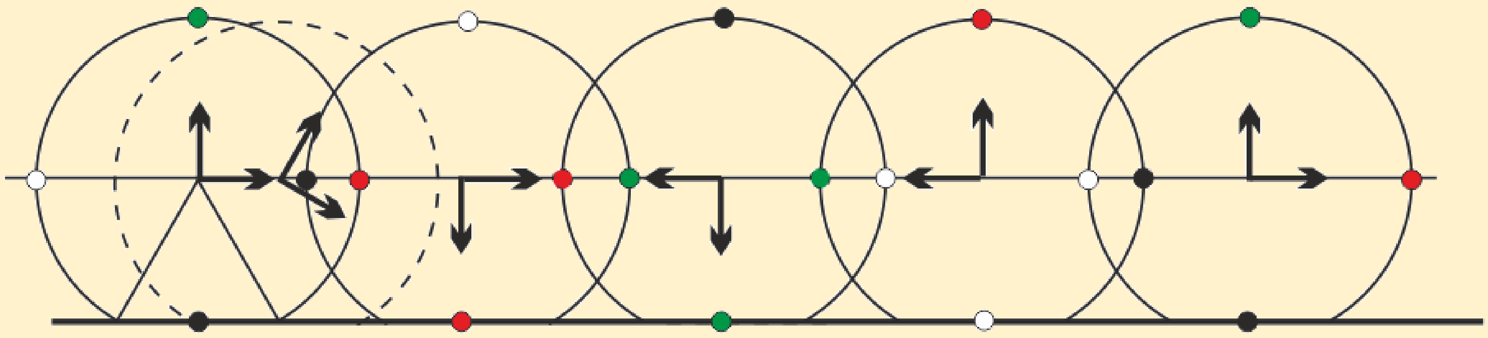
$c=\cos(2(k-a))$

$d=a-c+x-y$

$2(k-a)+(a-x)+y=a$

$0<y=x-2(k-a)=3x-2\arcsin(x)<x$

Kerék mozgása II (az ívhossz a forgatás függvénye)



$$2k=\pi$$

$$a=k\text{-}\arcsin(x)+x$$

Konklúzió

szög $\Rightarrow$ ívhossz

kapcsolat nem működik jól, ha a mozgás paramétere az ívhossz a fix görbén, akkor nem periodikus görbét kapunk, ha a mozgó görbe elfordulásával paraméterezünk, akkor a görbe két pontja közötti ív hossza függ az ív mozgó görbén elfoglalt helyzetétől.

Ötlet: Kapcsoljuk a szögmérést az ívhossz számításhoz.

Flexible motions in Minkowski plane

Generalized angle measure:

Definition 2. Let $\gamma \subseteq X$ be a closed Jordan curve which is starlike with respect to a point p of the interior of the region bounded by γ. An *angle measure with respect to such a Jordan curve* is a (normalized) Borel measure μ_γ on γ for which the following properties hold:

(a) $\mu_\gamma(\gamma) = 2\pi$;

(b) for any $q \in \gamma$ we have $\mu_\gamma(\{q\}) = 0$; and

(c) any non-degenerate arc of γ has positive measure.

Angle in normed spaces (in common with V.Balestra, H.Martini and R. Teixeira) *Aequationes Mathematicae* (2016)

Generalized rotation

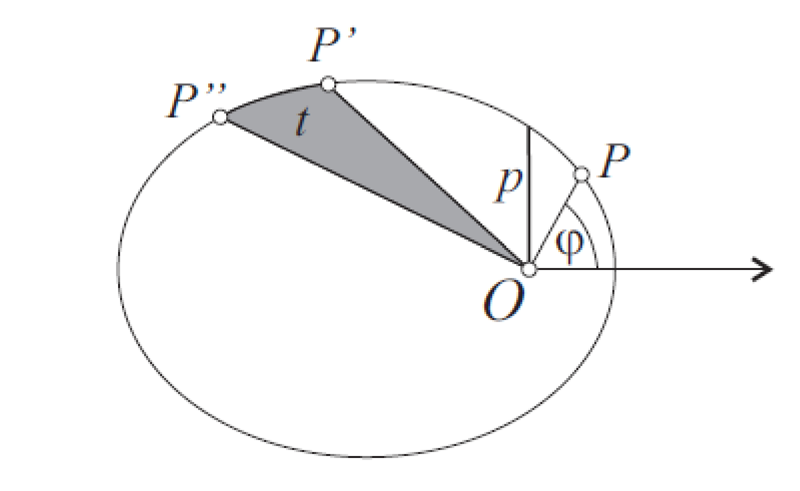
Definition 3. Let $(X, ||\cdot||)$ be a Minkowski plane and let γ be a closed Jordan curve which is starlike with respect to a point p of the interior of the region bounded by γ. Let $\mu_{\gamma,p}$ be a generalized angle measure as in the previous definition. A *general rotation* (with respect to $\mu_{\gamma,p}$) is a transform $\mathrm{rot}_{\mu_{\gamma,p}} : X \to X$ for which the following three properties hold:

(a) The transform $\mathrm{rot}_{\mu_{\gamma,p}}$ leaves invariant the pencil $\mathcal{R}(p)$ of rays with origin in p. In other words, if $r \subseteq X$ is a ray with origin p, then $\mathrm{rot}_{\mu_{\gamma,p}}(r)$ is also a ray with origin p.

(b) For each $\alpha > 0$, $\mathrm{rot}_{\mu_{\gamma,p}}$ leaves invariant the homothetic curve $\gamma_{\alpha,p} := p + \alpha(\gamma - p)$, i.e., for such a curve we have $\mathrm{rot}_{\mu_{\gamma,p}}(\gamma_{\alpha,p}) \subseteq \gamma_{\alpha,p}$.

(c) The function $r \in \mathcal{R}(p) \mapsto \mu_{\gamma,p}\left(\mathrm{rot}_{\mu_{\gamma,p}}(r), r\right)$ is constant. Intuitively, $\mathrm{rot}_{\mu_{\gamma,p}}$ "rotates every ray of $\mathcal{R}(p)$ by a same angle".

Area based general rotation (Kepler's law)



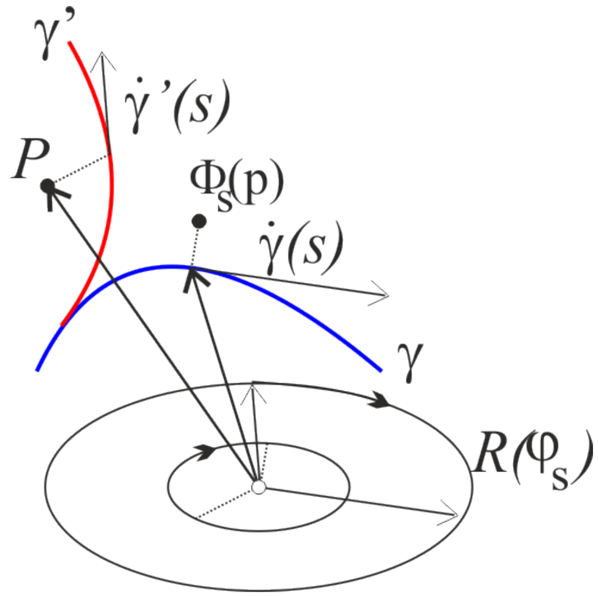
With respect to μ and G from above, for every real number $0 \leq t \leq 2\pi$ there is a generalized rotation of the Euclidean plane about O with this angle t. By Kepler's second law about planetary motions, the angle t of a generalized rotation is proportional to the time of the motion of the planet. Hence the generalized rotation with angle t maps the current position P' of the planet to that point P'' of the orbit where the planet arrives after time t.

Ruletta definíció:

Definition 7. *If the rectifiable Jordan curve $\gamma'(s)$ rolls, without slipping, on the rectifiable Jordan curve $\gamma(s)$, then we define* the flexible motion corresponding to the rolling curves γ *and* γ' *as the following set of mappings:*

$$\{\Phi_s(\mathbf{p}) = \gamma(s) + \mathrm{R}(\varphi_s)\,(\mathbf{p} - \gamma'(s)) : \; s \in [0, \beta]\}, \tag{6}$$

where $R(\varphi_s) \in \mathcal{R}(\partial B, \mu_l, o)$ denotes the general rotation which maps the (oriented) direction $\dot{\gamma}(s)$ to the (also oriented) direction $\dot{\gamma}'(s)$. A curve given by the graph of a fixed point $\mathbf{p} = P$ is called the roulette *of P.*



Second Euler- Savary equation

Theorem 3 (Second Euler-Savary equation). *If the unit circle of the Minkowski plane is two times continuously differentiable, then the following equality holds:*

$$\chi_\gamma - \chi_{\gamma'} = \frac{1}{r_\gamma} - \frac{1}{r_{\gamma'}} = \frac{\sigma(T_K)}{\sigma^2(t_K)}\frac{1}{\alpha_K}\,. \tag{12}$$

Here r_γ is the curvature radius of the fixed polode at its point $K = \gamma_s$, $r_{\gamma'}$ is the curvature radius of the moving polode at its point $K = \gamma'_s$, and α_K is the length of the common velocity vector of the fixed and moving polodes at the moment s and at the instantaneous pole $K = \gamma(s) = \gamma'(s)$.

First Euler-Savary equation

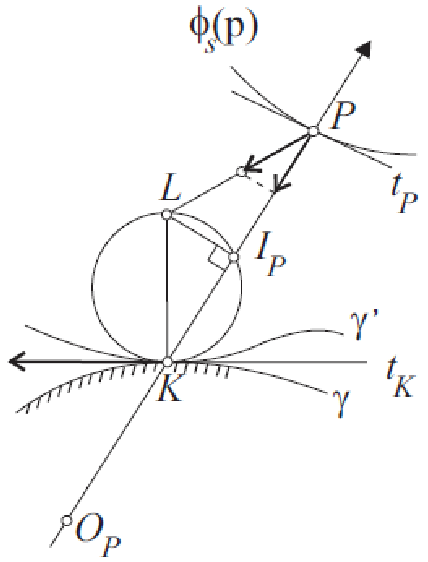
Theorem 4. *The instantaneous center K and the curvature center O_P of the roulette at its point $P \neq K$ satisfy the equality*

$$\|\overrightarrow{O_P P}\| = \frac{\|\overrightarrow{KP}\|^2}{\|\overrightarrow{I_P P}\|}, \tag{15}$$

where the second intersection point of the path normal line at P with the inflection curve is the point I_P.

By the law of sine introduced earlier, $O_P P$ and $I_P P$ are always marked off in the same orientation along the line KP. Thus, when I_P has been established, the orientation of $I_P P$ gives the orientation of $O_P P$. Hence equality (15) has an equivalent form for directed segments (with Minkowski lengths):

$$\frac{1}{KP} - \frac{1}{KO_P} = \frac{1}{KI_P}. \tag{16}$$



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Balestro, V., G. Horváth, Á., Martini, H.: Angle measures, general rotations, and roulettes in normed planes (in common with V. Balestro and H. Martini) *Analysis and Mathematical Physics* (2016) DOI: 10.1007/s13324-016-0155-3

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