Embedding of Incidence Structures into Projective Planes

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Incidence Structures

An incidence structure is a pair $(\mathcal{P}, \mathcal{L})$ where

- $\mathcal{P}$ is a finite non empty pointset;
- $\mathcal{L}$ is a family of subsets of $\mathcal{P}$, members of $\mathcal{L}$ are called lines;
- a point-line pair (P, ℓ) is called incident when $P \in \ell$;
- the number of lines incident with a point is constant,
- the number of points incident with a line is constant.

$PG(2, \mathbb{K})$:= projective plane over a field $\mathbb{K}$.
Embedding of $(\mathcal{P}, \mathcal{L})$ into $PG(2, \mathbb{K})$:= incidence preserving injective map $(\mathcal{P}, \mathcal{L}) \mapsto PG(2, \mathbb{K})$, i.e.

- three points of $(\mathcal{P}, \mathcal{L})$ are collinear if and only if they are mapped to three collinear points in $PG(2, \mathbb{K})$.

Trivial necessary embeddability condition:
If $(\mathcal{P}, \mathcal{L})$ can be embedded in $PG(2, \mathbb{K})$ then

- two distinct points of $(\mathcal{P}, \mathcal{L})$ are incident with at most one line;
- two distinct lines of $(\mathcal{P}, \mathcal{L})$ are incident with at most one point.

Line spaces

A line space is an incidence structure $(\mathcal{P}, \mathcal{L})$ where

- two distinct points of $(\mathcal{P}, \mathcal{L})$ are incident with exactly one line.

A dual line space is an incidence structure $(\mathcal{P}, \mathcal{L})$ where

- two distinct lines of $(\mathcal{P}, \mathcal{L})$ are incident with exactly one point;

Remark

Important incidence structures other than line spaces and their duals are configurations, such as Desargues, Pascal, and k-nets.

Remark

Certain line spaces and their duals are well known from classical Finite geometry.

- *Finite projective planes;*
- *Finite affine planes;*
- *Abstract unitals (more generally 2-block designs);*
- *Abstract ovals.*

Embedding of projective and affine planes into $PG(2, \mathbb{K})$

$(\mathcal{P}, \mathcal{L})$:=line space embeddable in $PG(2, \mathbb{K})$.

- If $(\mathcal{P}, \mathcal{L})$ is a projective plane then $(\mathcal{P}, \mathcal{L}) \cong PG(2, \mathbb{F})$ with $\mathbb{F}$ subfield of $\mathbb{K}$.
- If $(\mathcal{P}, \mathcal{L})$ is an affine plane then either $(\mathcal{P}, \mathcal{L}) \cong AG(2, \mathbb{F})$ with $\mathbb{F}$ subfield of $\mathbb{K}$, or it has order 3 and $\mathrm{char}(\mathbb{K}) \neq 3$.

Remark

In the exceptional case $(\mathcal{P}, \mathcal{L}) \cong AG(3, 2)$, the 9 points are mapped to the inflections of a nonsingular cubic of $PG(2, \mathbb{K})$.

Remark

In the exceptional case $(\mathcal{P}, \mathcal{L}) \cong AG(3, 2)$, embedding in $PG(2, q)$ is only possible for $q \equiv 1 \pmod 3$.

For the proofs, see G.K. Journal of Geometry 15 (1981), 170-174.

Unitals naturally embedded in $PG(2, \mathbb{K})$

A line space $(\mathcal{P}, \mathcal{L})$ is an *abstract unital* of order n if $|\mathcal{P}| = n^3 + 1, \; |\mathcal{L}| = n^2(n^2 - n + 1)$, while any point is incident with n^2 lines and any line is incident with $n + 1$ points.

Abstract unital of order q embedded in $PG(2, q^2)$:=unital in $PG(2, q^2)$

Definition

Set $\mathcal{H}$ of all isotropic points of unitary polarity of $PG(2, q^2)$ is a unital of order q, called the classical unital.

Remark

Equivalent definitions:

$\mathcal{H} := \{P = (a_1, a_2, a_3) \in PG(2, q^2) | a_1^{n+1} + a_2^{q+1} + a_3^{q+1} = 0\}$.

$\mathcal{H}$*:=set of all points in* $PG(2, q^2)$ *of the Hermitian curve with equation* $X_1^{n+1} + X_2^{q+1} + X_3^{q+1} = 0$.

Unitals naturally embedded in $PG(2, q^2)$, cont.

Definition

BM-unitals in $PG(2, q^2)$:=Unitals in $PG(2, q^2)$ which were constructed by Buekenhout (1974) using the four-dimensional Bruck-Barlotti representation of $AG(2, q^2)$.

Remark

- *Equations of BM-unitals (Ebert (1998)) are complicated.*
- *BM-unital is of Hirschfeld-Szőnyi type if it is union of q (pairwise hyperosculating) conics through a point of the unital.*
- *Every non Hirschfeld-Szőnyi type BM-unital is the image of the classical unital by a quadratic transformation of* $PG(2, q^2)$. *(Aguglia, G.K., Giuzzi 2003).*

Open question: Are there abstract unitals of order q (other than classical and BM-unitals) which can be embedded in $PG(2, q^2)$?

Unitals not naturally embedded in $PG(2, q^2)$

Problem: Is the natural one the unique embedding of the known unitals of order q in $PG(2, q^2)$?

Definition

Two embeddings of an abstract unital $\mathcal{U}$ of order q into $PG(2, q^2)$, say Φ and Ψ, are considered equivalent when $\Phi(\mathcal{U})$ can be transformed in $\Psi(\mathcal{U})$ by a collineation of $PG(2, q^2)$.

Theorem

(G.K., Siciliano, Szőnyi 2017) For the classical unital, the answer to the above Problem is yes.

Full points of the classical unital

In $PG(2, q^2)$, let $\mathcal{H}$ be the classical unital.

Definition

Let ℓ_1 and ℓ_2 be two distinct lines (chords) of $\mathcal{H}$. A point $P \in \mathcal{H}$ w.r.t. (ℓ_1, ℓ_2) is called full, if the projection from P of ℓ_1 on ℓ_2 is surjective. $f(\ell_1, \ell_2)$:= number of full points w.r.t. (ℓ_1, ℓ_2).

Theorem

(G.K., Siciliano, Szőnyi, Sonnino, 2016)
If $\ell_1 \cap \ell_2 \in \mathcal{H}$ *then* $f(\ell_1, \ell_2) = 0$.

- *If* q *is even then* $f(\ell_1, \ell_2) = 1$ *for any line* ℓ_2 *with* $\ell_1 \cap \ell_2 \notin \mathcal{H}$.
- *If* q *is odd then* $f(\ell_1, \ell_2) \in \{0, 2\}$. *For any* ℓ_1 *the number of lines* ℓ_2 *with* $f(\ell_1, \ell_2) = 2$ *is* $\approx$ *half of the lines of* $\mathcal{H}$ *missing* ℓ_1 *(*$\approx \frac{1}{2}(q^4 - 2q^3)$*)*

Projectivity group of a line, for q odd

Let q be odd.

For two lines ℓ_1, ℓ_2 of $\mathcal{H}$ with a full points P_1 and P_2, the product of the projection from P_1 of ℓ_1 to ℓ_2 with the projection from P_2 of ℓ_2 to ℓ_1 is called a *projectivity* of ℓ_1.

If ℓ_2 ranges over the set of lines with $f(\ell_1, \ell_2)$ then the arising projectivities generate a group, the *projectivity group* of ℓ_1.

Lemma

For any line ℓ_1 of $\mathcal{H}$, the projectivity group of ℓ is a cyclic group of order $q+1$ acting transitively on ℓ_1.

Projectivity group of a line, for q even

Let q be even.

Let ℓ_1, ℓ_2, ℓ_3 be three lines of $\mathcal{H}$ which are pairwise missing.
Let P_1, P_2, P_3 be the full points of ℓ_1, ℓ_2, ℓ_2, ℓ_3 and ℓ_3, ℓ_1, respectively.
The product of the projection from P_1 of ℓ_1 to ℓ_2 with the projection from P_2 of ℓ_2 to ℓ_1, and with the projection ℓ_3 to ℓ_1 is called a *projectivity* of ℓ_1.

If ℓ_2, ℓ_3 range over the set of lines then the arising projectivities generate a group, the *projectivity group* of ℓ_1.

Lemma

For any line ℓ_1 of $\mathcal{H}$, the projectivity group of ℓ is a cyclic group of order $q+1$ acting transitively on ℓ_1.

Projectivity groups of unitals embedded in $PG(2, q^2)$

Let $\mathcal{U}$ be a unital of order q which is embedded in $PG(2, q^2)$.

Lemma

If the projectivity group of a line ℓ_1 of $\mathcal{U}$ is a cyclic group acting transitively on ℓ_1 then ℓ_1 is a Baer subline of $PG(2, q^2)$.

Remark

The proof of Lemma depends on the classification of subgroups of $PGL(2, q^2)$.

Corollary

If $\mathcal{U}$ is isomorphic to $\mathcal{H}$, as abstract unitals, and $\mathcal{U}$ is embedded in $PG(2, q^2)$ then $\mathcal{U}$ is the classical unital of $PG(2, q^2)$.

Abstract oval from a projective oval

Π:=Projective plane of order n;
Ω:=Oval in Π, i.e.

- no three points of Ω is collinear,
- at each point $P \in \Omega$ there exists a unique line which meets Ω on in P, (called the tangent to Ω at P).

Ambient of Ω:=$(\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ is the pointset of Π, and $\mathcal{L}$ is the set of all tangents of and secants of Ω.
The ambient of Ω is a dual line space.

Remark

From every point $P \in \Pi \setminus \Omega$, one can project Ω onto itself. This projection is called an involution of Ω.
The set of all involutions is a quasi 2-transitive set $\mathcal{F}$ of involutory permutations on Ω (i.e. for any two point pairs (P_1, P_2) and (Q_1, Q_2) on Ω such that $P_i \neq Q_j$ with $1 \leq i, j \leq 2$ there exists a unique $f \in \mathcal{F}$ such that $f(P_1) = P_2, f(Q_1) = Q_2)$.

Abstract ovals

Let $(\Omega, \mathcal{F})$ be a quasi 2-transitive set of involutory permutations on a set Ω of length $n+1$.

- secant of $(\Omega, \mathcal{F})$:=two distinct points $A, B \in \Omega$ together with all involutions $f \in \mathcal{F}$ s.t. $f(A) = B$.
- tangent of $(\Omega, \mathcal{F})$:=a point $P \in \Omega$ together with all involutions $f \in \mathcal{F}$ s.t. $f(P) = P$.

$(\mathcal{P}, \mathcal{L})$:=dual line space where
$\mathcal{P} := \Omega \cup \mathcal{F}$ and $\mathcal{L}$:=$\mathcal{T} \cup \mathcal{S}$ with
$\mathcal{T}$ and $\mathcal{S}$ are the set of all tangents and secants, respectively.

Definition

The dual line space $(\mathcal{P}, \mathcal{L})$ is called and an *abstract oval* of order n.

There exist abstract ovals of order 8 which are not projective, but do not for $n < 8, n = 9, 10$.
Open problem: Are there abstract ovals of order $n > 10$ which are not projective?

Embedding of abstract ovals in $PG(2, q)$

Let $(\mathcal{P}, \mathcal{L})$ be an abstract oval of order n which can be embedded in $PG(2, q)$.

Remark

$n \leq q$ and if equality holds then $(\mathcal{P}, \mathcal{L})$ is projective.

Theorem

(G.K., Faina 1981) If q is odd then $q = n^r$ and $(\mathcal{P}, \mathcal{L})$ is mapped to a subplane $PG(2, n)$ of $PG(2, q)$. In particular, $(\mathcal{P}, \mathcal{L})$ is projective.

Theorem

(L.M. Abatangelo, G. Raguso 1981) If q is even and Ω is mapped into a conic then $q = n^r$ and $(\mathcal{P}, \mathcal{L})$ is mapped to a subplane $PG(2, n)$ of $PG(2, q)$. In particular, $(\mathcal{P}, \mathcal{L})$ is projective.

Proofs depend on Segre's lemma. Open problem: Can be dropped the extra hypothesis for q even?