Változatok a Minkowski problémára

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Reconstruction of smooth closed convex surfaces from Gauss curvature

- X is a compact C^2_+ hypersurface in $\mathbb{R}^n$
- u_x is exterior unit normal at $x \in X$
- $\kappa_X(u_x) > 0$ is the Gauss curvature

Observation (Minkowski)

$$\int_{S^{n-1}} u \cdot \kappa_X(u)^{-1}\, du = o. \tag{1}$$

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Minkowski problem (E.g. Inverse problem of short wave diffraction)
For continuous $\kappa : S^{n-1} \to \mathbb{R}_+$ satisfying (1), find C^2_+ hypersurface $X \subset \mathbb{R}^n$ such that $\kappa(u_x)$ is the Gauss curvature at $x \in X$.

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Monge-Ampere type differential equation on S^{n-1}:

$$\det(\nabla^2 h + h\, I) = \kappa^{-1}$$

where $h(u) = \max\{\langle u, x\rangle : x \in X\}$ is the support function.

Notation

- K, C - convex bodies in $\mathbb{R}^n$
  (convex compact with non-empty interior)
- $V(K)$ - volume (Lebesgue measure)
- $\mathcal{H}^{n-1}$ - $(n-1)$-Hausdorff measure
- h_K - support function of K
  $h_K(u) = \max\{\langle u, x\rangle : x \in K\}$ for $u \in \mathbb{R}^n$
- L - linear subspace, $L \neq \{o\}, \mathbb{R}^n$
- μ - non-trivial Borel measure on S^{n-1}

Surface area measure

S_K - surface area measure of K on S^{n-1}

- $\nu_K(x) = \{u \in S^{n-1} : h_K(u) = \langle x, u \rangle\}$ for $x \in \partial K$ (all possible exterior unit normals at x)
- For $\Xi \subset \partial K$, $S_K(\nu_K(\Xi)) = \mathcal{H}^{n-1}(\Xi)$

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- ∂K is C^2_+ and $\kappa(u_x)$ Gauss curvature at $x \in \partial K$, $u_x = \nu_K(x)$,

$$S_K(\omega) = \int_\omega \kappa^{-1}(u)\, du \quad \text{for } \omega \subset S^{n-1}.$$

- K polytope, $F_1, \ldots, F_k$ facets, u_i exterior unit normal at F_i

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Properties

- $S_K(S^{n-1})$ =surface area of K
- $V(K) = \frac{1}{n} \int_{S^{n-1}} h_K \, dS_K$

Minkowski problem

"Minkowski problem" (Minkowski, Alexandrov, Nirenberg)
$\mu = S_K$ for some unique convex body K (up to translation) iff

1. $\mu(L \cap S^{n-1}) < \mu(S^{n-1})$ for any L with $\dim L = n-1$
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To solve the Minkowski problem,

- Minimize $\int_{S^{n-1}} h_C \, d\mu$ under the condition $V(C) = 1$
- Uniqueness comes from uniqueness in the Minkowski inequality

Brunn-Minkowski inequality

$$\begin{aligned}\alpha K + \beta C &= \{\alpha x + \beta y : x \in K,\ y \in C\} \\ &= \{x \in \mathbb{R}^n : \langle u, x\rangle \leq \alpha h_K(u) + \beta h_C(u)\ \forall u \in S^{n-1}\}\end{aligned}$$

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Brunn-Minkowski inequality $\alpha, \beta > 0$

$$V(\alpha\, K + \beta\, C)^{\frac{1}{n}} \geq \alpha\, V(K)^{\frac{1}{n}} + \beta\, V(C)^{\frac{1}{n}}$$

with equality iff K and C are homothetic.
Remark Yields the isoperimetric inequality if C is the unit ball

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Minkowski inequality If $V(K) = V(C)$, then

$$\int_{S^{n-1}} h_C\, dS_K \geq \int_{S^{n-1}} h_K\, dS_K,$$

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Logarithmic Minkowski problem - Cone volume measure

$dV_K = \frac{1}{n} h_K dS_K$ - cone volume measure on S^{n-1} if $o \in \mathrm{int} K$ (Gromov, Milman, 1986)

- K polytope, $F_1, \ldots, F_k$ facets, u_i exterior unit normal at F_i

$$V_K(\{u_i\}) = \frac{h_K(u_i)\mathcal{H}^{n-1}(F_i)}{n} = V(\mathrm{conv}\{o, F_i\}).$$

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$$h \det(\nabla^2 h + h I) = f$$

Even cone volume measures

Theorem (B, Lutwak, Yang, Zhang)

Let μ be an even Borel measure on S^{n-1}.
$\mu = V_K$ for some o-symmetric convex body K iff

(i) $\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \mu(S^{n-1})$ *for any* $L \neq \{o\}, \mathbb{R}^n$

(ii) *If equality holds for some L, then* $\operatorname{supp} \mu \subset L \cup L'$ *for some complementary L'*

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Conjecture (Uniqueness)

$V_K = V_C$ for o-symmetric convex bodies K and C with $V(K) = V(C)$ iff K and C have dilated direct summands; namely, $K = K_1 \oplus \ldots \oplus K_m$ and $C = C_1 \oplus \ldots \oplus C_m$ with $K_i = \lambda_i C_i$ for $\lambda_1, \ldots, \lambda_m > 0$.

Cone volume measure for certain non-centrally symmetric bodies

Theorem (B, Henk, Linke)

If the centroid of K is the origin, then

(i) $V_K(L \cap S^{n-1}) \leq \frac{\dim L}{n} \cdot V(K)$ *for any* $L \neq \{o\}, \mathbb{R}^n$

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Theorem (Zhu)

If μ is a discrete Borel measure on S^{n-1} such that any different $u_1, \ldots, u_n \in \operatorname{supp} \mu$ are independent, then $\mu = V_K$ for some polytope K

Isotropic position of a measure on S^{n-1}

Theorem (BLYZ)

Let μ be a Borel probability measure on S^{n-1}. There exists $A \in \mathrm{GL}(n)$ such that

$$\int_{S^{n-1}} \frac{Au}{\|Au\|} \otimes \frac{Au}{\|Au\|} \, d\mu(u) = \tfrac{1}{n} \, \mathrm{Id}_n$$

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- Sufficiency if $\mu(L \cap S^{n-1}) < \frac{\dim L}{n}$ is due to Klartag (supergaussian marginals of probability measures on $\mathbb{R}^n$)
- Discrete case is due to Carlen-Lieb-Loss (extremals for the Brascamp-Lieb inequality). See also Benneth&Carbery&Christ&Tao, Carlen&Cordero-Erausquin

Logarithmic Brunn-Minkowski inequality

$\alpha \in [0,1]$, $o \in \mathrm{int}\, K, \mathrm{int}\, C$

$$\alpha K +_0 (1-\alpha) C = \{x \in \mathbb{R}^n : \langle u, x \rangle \leq h_K(u)^\alpha h_C(u)^{1-\alpha} \ \forall u \in S^{n-1}\}$$

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Conjecture (Logarithmic Brunn-Minkowski conjecture)

$\alpha \in (0,1)$, K, C are o-symmetric

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with equality iff K and C have dilated direct summands.

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- Holds for the volume in $\mathbb{R}^{2n} = \mathbb{C}^n$ if K and C are complex convex bodies (Rotem)

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- Holds for the volume in $\mathbb{R}^{2n} = \mathbb{C}^n$ if K and C are complex convex bodies (Rotem)
- If it holds for the volume, it does hold for any log-concave measure (Saroglou)

Alexandrov's problem

Alexandrov's Integral Curvature, 1940

For $o \in \mathrm{int} K$ and $\omega \subset S^{n-1}$,

$$C_K(\omega) = \mathcal{H}^{n-1}(\nu_K \circ r_K(\omega))$$

where for $u \in S^{n-1}$, $r_K(u) = \varrho_K(u)u \in \partial K$ for $\varrho_K(u) > 0$.

Theorem (Alexandrov)

For a finite Borel measure μ on S^{n-1}, $\mu = C_K$ if and only if

- $\mu(S^{n-1}) = \mathcal{H}^{n-1}(S^{n-1})$,
- *for any proper closed convex $\omega \subset S^{n-1}$, we have*

$$\mu(S^{n-1} \backslash \omega) > \mathcal{H}^{n-1}(\omega^*).$$

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Theorem (B,Yiming, Zhang, 2016)

There is a proof where one minimizes

$$\int_{S^{n-1}} \log h_K(u)\, d\mu(u) - \int_{S^{n-1}} \log \varrho_K(u)\, du.$$

L_p surface area measures

L_p surface area measures (Firey, Lutwak 1990) $p \in \mathbb{R}$

$$dS_{K,p} = h_K^{1-p}\, dS_K = n h_K^{-p}\, dV_K$$

Examples

- $S_{K,1} = S_K$
- $S_{K,0} = nV_K$
- $S_{K,-n}$ related to the $\mathrm{SL}(n)$ invariant curvature $\frac{\kappa_K(u)}{h_K(u)^{n+1}}$

Theorem (Chou-Wang (2005), Hug-LYZ (2006))

If $p > 1$, $p \neq n$, then any finite Borel measure μ on S^{n-1} not concentrated on any closed hemisphere is of the form $\mu = S_{K,p}$.

Theorem (Zhu (2015))

If $p < 1$, then any discrete measure μ on S^{n-1} not concentrated on any closed hemisphere whose support is in general position is of the form $\mu = S_{K,p}$.

Differential equation for L_p surface area measures

$$h^{1-p}\det(\nabla^2 h + hI) = f$$

Theorem (Chou-Wang (2005))

If $-n < p < 1$, f is bounded and its infimum is positive, then $f\, d\mathcal{H}^{n-1} = S_{K,p}$.

Remak There must be conditions on f if $p = -n$

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Theorem (B-Trinh (2017))

If $n = 2$*,* $0 < p < 1$*, and* f *is bounded, then* $f\,d\mathcal{H}^{n-1} = S_{K,p}$*.*

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General Ideas to solve L_p-Minkowski problem for given $p > 1$, μ

- Minimize $\int_{S^{n-1}} h_K^p\,d\mu$ under the condition $V(K) = 1$
- Weak approximation by discrete measures (polytopes)

Dual curvature measures

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Huang, Lutwak, Yang, Zhang, 2016 (Acta Mathematica)

$$\widetilde{C}_{K,q}(\nu_K \circ r_K(\omega)) = \int_\omega \varrho^q(u)\,du \quad \text{for } \omega \subset S^{n-1}$$

Most interesting if $q \in [0, n]$

Examples

- $\widetilde{C}_{K,0} = C_{K^*}$ (Alexandrov's Integral Curvature)
- $\widetilde{C}_{K,n} = nV_K$ (cone volume measure)

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$$\widetilde{C}_{K,q}(\nu_K \circ r_K(\omega)) = \int_\omega \varrho^q(u)\, du \qquad \text{for } \omega \subset S^{n-1}$$

Most interesting if $q \in [0, n]$

Examples

- $\widetilde{C}_{K,0} = C_{K^*}$ (Alexandrov's Integral Curvature)
- $\widetilde{C}_{K,n} = nV_K$ (cone volume measure)

Theorem (Zhao (2016), B-Henk-Hassen (2016))

$0 < q < n$, and μ is finite even non-trival Borel measure on S^{n-1}. Then $\mu = \widetilde{C}_{K,q}$ for o-symmetric K iff for every non-trivial L,

$$\mu(L \cap S^{n-1}) < \frac{\dim L}{q} \cdot \mu(S^{n-1}).$$

L_p Dual curvature measures

Lutwak, Yang, Zhang, 2016 (manuscript), $p, q \in \mathbb{R}$

$$d\widetilde{C}_{K,p,q} = h_K^{-p}\, d\widetilde{C}_{K,q}$$

Examples

- $\widetilde{C}_{K,p,n} = S_{K,p}$
- $\widetilde{C}_{K,0,q} = \widetilde{C}_{K,q}$

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Dual Intrinsic Volume $q > 0$

$$V_q(K) = \frac{1}{n} \int_{S^{n-1}} \varrho_K^q \, d\mathcal{H}^{n-1}$$

An idea to solve L_p-dual Minkowski problem for $p, q > 1$, μ

- Minimize $\int_{S^{n-1}} h_K^p \, d\mu$ under the condition $V_q(K) = 1$
- Weak approximation by discrete measures (polytopes)