On the covering index of convex bodies

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1 Introduction

Let $\mathbb{E}^d$ denote the d-dimensional Euclidean space. A d-dimensional convex body K is a compact convex subset of $\mathbb{E}^d$ with nonempty interior. Moreover, K is *o-symmetric* if $K = -K$. The *Minkowski sum* or simply the *vector sum* of two convex bodies $K, L \subseteq \mathbb{E}^d$ is defined by

$$K + L = \{k + l : k \in K, l \in L\}.$$

A *homothetic copy*, or simply a *homothet*, of K is a set of the form $M = \lambda K + x$, where λ is a nonzero real number and $x \in \mathbb{E}^d$. If $\lambda > 0$, then M is said to be a *positive homothet* and if in addition, $\lambda < 1$, we have a *smaller positive homothet* of K. Let C^d denote a d-dimensional cube, B^d a d-dimensional ball, Δ^d a d-simplex and ℓ a line segment (or more precisely, an affine image of any of these convex bodies). We use the symbol $\mathscr{K}^d$ for the metric space of d-dimensional convex bodies under the (multiplicative) Banach-Mazur distance $d_{BM}(\cdot,\cdot)$. That is, for any $K, L \in \mathscr{K}^d$,

$$d_{BM}(K,L) = \inf\{\delta \geq 1 : a \in K, b \in L, L - b \subseteq T(K - a) \subseteq \delta(L - b)\},$$

where the infimum is taken over all invertible linear operators $T : \mathbb{E}^d \longrightarrow \mathbb{E}^d$ [29].

1.1 The Boltyanski-Hadwiger illumination conjecture (1960)

The famous Hadwiger Covering Conjecture [13,14,20] – also called the Gohberg-Markus-Hadwiger Covering Conjecture – states that any $K \in \mathscr{K}^d$, can be covered by 2^d of its smaller positive homothetic copies with 2^d homothets needed only if K is an affine d-cube. This conjecture appears in several equivalent forms one of which we discuss here. Boltyanski [7] and Hadwiger [15] introduced two notions of illumination of a convex body, the former being '*illumination by directions*' while the latter being '*illumination by points*'. The two notions are actually equivalent [7] and K is said to be *illuminated* if all points on the boundary of K are illuminated (in either sense). The *illumination number* $I(K)$ of K is the smallest n for which K can be illuminated by n points/directions. Furthermore, Boltyanski [7,8] showed that $I(K) = n$ if and only if the smallest number of smaller positive homothets of K that can cover K is n. Thus the Hadwiger Covering Conjecture can be reformulated as the Boltyanski-Hadwiger Illumination Conjecture, which states that for any d-dimensional convex body K we have $I(K) \leq 2^d$, and $I(K) = 2^d$ only if K is an affine d-cube.

1.2 The illumination parameter

For instance, it can be seen that in the definition of illumination number $I(K)$, the light sources can be taken arbitrarily far from K. However, it seems natural to start with a relatively small number of light sources and quantify how far they need to be from K in order to illuminate it. This is the idea behind the illumination parameter $\mathrm{ill}(K)$ of an o-symmetric convex body K defined by the first named author [3] as follows.

$$\mathrm{ill}(K) = \inf\left\{\sum_i \|p_i\|_K : \{p_i\} \text{ illuminates } K, p_i \in \mathbb{E}^d\right\},$$

where $\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$ is the norm of $x \in \mathbb{E}^d$ generated by the symmetric convex body K. Clearly, $I(K) \leq \mathrm{ill}(K)$, for o-symmetric convex bodies. Several authors have investigated the illumination parameter of o-symmetric convex bodies [3,6,16,22] and related ideas such as the vertex index [4], determining exact values in several cases.

1.3 The covering parameter

Inspired by the above quantification ideas, Swanepoel [30] introduced the covering parameter of a d-dimensional convex body to quantify its covering properties. This is given by

$$C(K) = \inf\left\{\sum_i (1-\lambda_i)^{-1} : K \subseteq \bigcup_i (\lambda_i K + t_i), 0 < \lambda_i < 1, t_i \in \mathbb{E}^d\right\}.$$

Thus large homothets are penalized in the same way as far away light sources are penalized in the definition of illumination parameter. Note here K is not assumed to have any symmetry as the definition of covering parameter does not make use of the norm $\|\cdot\|_K$. In the same paper, Swanepoel obtained the following Rogers-type upper bounds on $C(K)$ when $d \geq 2$.

$$C(K) = \begin{cases} O(2^d d^2 \ln d), & \text{if } K \text{ is } o\text{-symmetric}, \\ O(4^d d^{3/2} \ln d), & \text{otherwise.} \end{cases} \tag{1}$$

He further showed that if K is o-symmetric, then

$$\mathrm{ill}(K) \leq 2C(K). \tag{2}$$

2 The covering index
2.1 Definition

Given a positive integer m, Lassak [17] introduced the *m-covering number* of a convex body K as the minimal positive homothety ratio needed to cover K by m homothets. That is,

$$\gamma_m(K) = \inf\left\{\lambda > 0 : K \subseteq \bigcup_{i=1}^{m}(\lambda K + t_i), t_i \in \mathbb{E}^d, i = 1, \ldots, m\right\}.$$

Definition 1 Let K be a d-dimensional convex body. We define the *covering index* of K as

$$\mathrm{coin}(K) = \inf\left\{\frac{m}{1-\gamma_m(K)} : \gamma_m(K) \leq 1/2, m \in \mathbb{N}\right\}.$$

Intuitively, $\mathrm{coin}(K)$ measures how K can be covered by a relatively small number of positive homothets all corresponding to the same relatively small homothety ratio. We note that $\mathrm{coin}(K)$ is an affine invariant quantity assigned to K, i.e., if $A : \mathbb{E}^d \longrightarrow \mathbb{E}^d$ is an invertible linear map then $\mathrm{coin}(A(K)) = \mathrm{coin}(K)$. The reader may be a bit surprised to see the restriction $\gamma_m(K) \leq 1/2$. One

However, there are other far more compelling reasons for choosing $1/2$ as the threshold. To understand these better, we define

$$f_m(K) = \begin{cases} \dfrac{m}{1-\gamma_m(K)}, & \text{if } 0 < \gamma_m(K) \leq \dfrac{1}{2}, \\ +\infty, & \text{if } \dfrac{1}{2} < \gamma_m(K) \leq 1. \end{cases}$$

Thus $\operatorname{coin}(K) = \inf\{f_m(K) : m \in \mathbb{N}\}$. Later in Theorem 1, we show that for any $K, L \in \mathscr{K}^d$ and $m \in \mathbb{N}$ such that $\gamma_m(K) \leq 1/2$ and $\gamma_m(L) \leq 1/2$,

$$f_m(K) \leq d_{BM}(K,L) f_m(L), \tag{3}$$

and

$$f_m(K) \geq \frac{d_{BM}(K,L)}{2d_{BM}(K,L)-1} f_m(L), \tag{4}$$

establishing a strong connection with the Banach-Mazur distance of convex bodies. The proofs of relations (3) and (4) make extensive use of homothety ratios to be less than or equal to half. This shows that the 'half constraint' in the definition of covering index results in a quantity with potentially nicer properties. In particular, relation (3) is important as for each m, it implies Lipschitz continuity of f_m on the subspace

$$\mathscr{K}_m^d := \left\{K \in \mathscr{K}^d : \gamma_m(K) \leq 1/2\right\}. \tag{5}$$

2.2 Relationship with other problems

Proposition 2 *For any o-symmetric d-dimensional convex body K,*

$$\mathrm{vein}(K) \le \mathrm{ill}(K) \le 2C(K) \le 2\,\mathrm{coin}(K),$$

and in general for $K \in \mathscr{K}^d$,

$$I(K) \le C(K) \le \mathrm{coin}(K).$$

Proposition 2 follows immediately from the definition of coin, the relation (2) and the observation

$$\begin{aligned}\mathrm{coin}(K) &= \inf\left\{\frac{m}{1-\gamma_m(K)} : \gamma_m(K) \le 1/2, m \in \mathbb{N}\right\}\\ &= \inf\left\{\frac{m}{1-\lambda} : K \subseteq \bigcup_{i=1}^{m}(\lambda K + t_i), 0 < \lambda \le 1/2, t_i \in \mathbb{E}^d, m \in \mathbb{N}\right\}\\ &\ge C(K).\end{aligned}$$

2.3 Upper bounds

Proposition 3 (Rogers-type bounds) *Given* $K \in \mathscr{K}^d$, $d \geq 2$, *we have*

$$\operatorname{coin}(K) < \begin{cases} 2^{2d+1} d(\ln d + \ln\ln d + 5) = O(4^d d \ln d), & \text{if } K \text{ is } o\text{-symmetric,} \\ 2^{d+1}\binom{2d}{d} d(\ln d + \ln\ln d + 5) = O(8^d d \ln d), & \text{otherwise.} \end{cases}$$

Proof Consider the covering of K by homothets $\frac{1}{2}K + t_i$, for some $t_i \in \mathbb{E}^d$, $i = 1, \ldots, m$. By (6), we have

$$m \leq \frac{\operatorname{vol}\left(K - \frac{1}{2}K\right)}{\operatorname{vol}\left(\frac{1}{2}K\right)} \,\theta\left(\frac{1}{2}K\right) = \frac{\operatorname{vol}\left(K - \frac{1}{2}K\right)}{\operatorname{vol}\left(\frac{1}{2}K\right)} \,\theta(K) < 2^d \frac{\operatorname{vol}(K-K)}{\operatorname{vol}(K)} \,\theta(K).$$

By the Rogers-Shephard inequality [26], $\frac{\operatorname{vol}(K-K)}{\operatorname{vol}(K)} \leq \binom{2d}{d}$. Note also that if K is o-symmetric, then $\frac{\operatorname{vol}(K-K)}{\operatorname{vol}(K)} = 2^d$. Finally, recall the Rogers' upper bound [25], $\theta(K) \leq d(\ln d + \ln\ln d + 5)$, for $d \geq 2$. The upper bounds of Proposition 3 follow. □

3 Properties of the covering index
3.1 Monotonicity

Lemma 1 (Monotonicity) *Let $l < m$ be positive integers. Then for any d-dimensional convex body K the inequality $f_l(K) > f_m(K)$ implies $m < f_l(K)$.*

Proof By assumption, $f_l(K) > f_m(K) = \frac{m}{1-\gamma_m(K)}$. On the other hand, $\frac{m}{1-\gamma_m(K)} > m$, completing the proof. □

This shows that for a fixed convex body K, $f_m(K)$ satisfies a special type of monotonicity with respect to m and as a result the covering index of any convex body can be obtained by calculating a finite minimum, rather than the infimum of an infinite set. In particular, if $f_l(K) < \infty$ for some l, then $\text{coin}(K) = \min\{f_m(K) : m < f_l(K)\}$.

3.2 Continuity

Theorem 1 (Continuity) *Let d be any positive integer.*
(i) For any $K, L \in \mathscr{K}_m^d$, the relations (3) and (4) hold. Moreover, equality holds in (3) if and only if $d_{BM}(K,L) = 1$, i.e., L is an affine image of K and equality in (4) holds if and only if either $d_{BM}(K,L) = 1$ or $d_{BM}(K,L) > 1$ with

$$\gamma_m(K) = \frac{\gamma_m(L)}{d_{BM}(K,L)} = \frac{1}{2d_{BM}(K,L)}.$$

(ii) The functional $f_m : \mathscr{K}_m^d \longrightarrow \mathbb{R}$ is Lipschitz continuous with $|f_m(K) - f_m(L)| < d_{BM}(K,L)$, for all $K, L \in \mathscr{K}_m^d$. On the other hand, $f_m : \mathscr{K}^d \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, for all d and m.
(iii) Define $I_K = \{i : \gamma_i(K) \leq 1/2\} = \{i : K \in \mathscr{K}_i^d\}$, for any d-dimensional convex body K. If $I_L \subseteq I_K$, for some $K, L \in \mathscr{K}^d$, then

$$\operatorname{coin}(K) \leq \frac{2d_{BM}(K,L) - 1}{d_{BM}(K,L)} \operatorname{coin}(L) \leq d_{BM}(K,L) \operatorname{coin}(L). \tag{7}$$

(iv) The functional $\operatorname{coin} : \mathscr{K}^d \longrightarrow \mathbb{R}$ is lower semicontinuous for all d.
(v) Define

$$\mathscr{K}^{d*} := \left\{ K \in \mathscr{K}^d : \gamma_m(K) \neq 1/2, m \in \mathbb{N} \right\}.$$

Then the functional $\operatorname{coin} : \mathscr{K}^{d*} \longrightarrow \mathbb{R}$ is continuous for all d.

Proof of (i) - Step 1:

Proof (i) We first show

Proposition 4 *For any $K, L \in \mathscr{K}^d$,*

$$\gamma_m(K) \leq d_{BM}(K,L)\gamma_m(L) \tag{8}$$

holds and so γ_m is Lipschitz continuous on $\mathscr{K}^d$ with $|\gamma_m(K) - \gamma_m(L)| < d_{BM}(K,L)$, for all $K, L \in \mathscr{K}^d$.

Proof Let $\delta > 1$ be such that $d_{BM}(K,L) < \delta$. Now let $a \in K$, $b \in L$ and the invertible linear operator $T : \mathbb{E}^d \longrightarrow \mathbb{E}^d$ satisfy $L - b \subseteq T(K-a) \subseteq \delta(L-b)$. Moreover, let $\left\{\lambda L + x_i : x_i \in \mathbb{E}^d, i = 1, \ldots, m\right\}$ be a homothetic cover of L, having m homothets with homothety ratio $\lambda > 0$. Then

$$\begin{aligned} T(K-a) &\subseteq \delta(L-b) \subseteq \delta\left(\bigcup_{i=1}^{m}(\lambda L + x_i - b)\right) = \delta\left(\bigcup_{i=1}^{m}(\lambda(L-b) + x_i + (\lambda-1)b)\right) \\ &\subseteq \delta\left(\bigcup_{i=1}^{m}(\lambda T(K-a) + x_i + (\lambda-1)b)\right) = \left(\bigcup_{i=1}^{m}(\delta\lambda T(K-a) + \delta x_i + \delta(\lambda-1)b)\right), \end{aligned}$$

which implies that there is a homothetic cover of $T(K-a)$ having m homothets with homothety ratio $\delta\lambda$. Hence there is a homothetic cover of K having m homothets with homothety ratio $\delta\lambda$. This implies that $\gamma_m(K) \leq \delta\gamma_m(L)$. Therefore, by taking $\inf \delta = d_{BM}(K,L)$, we get $\gamma_m(K) \leq d_{BM}(K,L)\gamma_m(L)$.

On the other hand, $\gamma_m(K) \leq 1$, $\gamma_m(L) \leq 1$, and (8) imply in a straightforward way that

$$|\gamma_m(K) - \gamma_m(L)| \leq d_{BM}(K,L)\left(1 - \frac{1}{d_{BM}(K,L)}\right) < d_{BM}(K,L),$$

whenever, $d_{BM}(K,L) > 1$, finishing the proof of Proposition 4. □

A consequence of the continuity of the m-covering number

The continuity of the m-covering number has some interesting consequences. For instance, it can be used to prove the following statement that settles Problem 6, posed by Brass, Moser and Pach, in Section 3.2 of [10]. We note that this statement was first proved in [24] via showing the upper semicontinuity of $\gamma_{H_d} : \mathscr{K}^d \longrightarrow \mathbb{R}$. Since we use the continuity of γ_m, our proof is simpler.

Proposition 1 (Problem 6, Section 3.2 [10]) *Let H_d denote the smallest number h for which every d-dimensional convex body can be covered by h smaller positive homothetic copies of itself. Let $\overline{H_d}$ be the smallest h for which there exists a positive $\lambda_d < 1$ such that every d-dimensional convex body can be covered by at most h of its homothetic copies with homothety ratio at most λ_d. Then $H_d = \overline{H_d}$ for every d.*

Proof Clearly, $H_d \leq \overline{H_d}$. On the other hand, as the space $\mathscr{K}^d$ is compact under the Banach-Mazur metric [21,32], therefore $\gamma_{H_d}(\mathscr{K}^d) \subseteq (0,1)$ is compact as well.Thus, there is a constant $c < 1$ such that $\gamma_{H_d}(K) \leq c$, for any $K \in \mathscr{K}^d$. As a result, we get that $\overline{H_d} \leq H_d$, finishing the proof of $\overline{H_d} = H_d$. $\square$

Proof of (i) – Step 2:

We now return to the main proof. To prove (3) let $K, L \in \mathscr{K}_m^d$. If $\gamma_m(K) \le \gamma_m(L)$, then $f_m(K) \le f_m(L) \le d_{BM}(K,L) f_m(L)$, with equality if and only if $d_{BM}(K,L) = 1$. Therefore, we can assume without loss of generality that $\gamma_m(K) > \gamma_m(L)$. Note that this together with $\gamma_m(K) \le 1/2$ and $\gamma_m(L) \le 1/2$ implies

$$\gamma_m(K) - (\gamma_m(K))^2 > \gamma_m(L) - (\gamma_m(L))^2. \tag{9}$$

Thus by using (8),

$$\frac{f_m(K)}{f_m(L)} = \frac{1-\gamma_m(L)}{1-\gamma_m(K)} < \frac{\gamma_m(K)}{\gamma_m(L)} \le d_{BM}(K,L),$$

which gives (3). In addition, equality never holds in this case. Thus equality in (3) holds if and only if $d_{BM}(K,L) = 1$.

Now to prove (4), we again use (8).

$$\begin{aligned} f_m(K) &= \frac{m}{1-\gamma_m(K)} \ge \frac{m}{1 - \frac{\gamma_m(L)}{d_{BM}(K,L)}} \\ &= \frac{d_{BM}(K,L)(1-\gamma_m(L))}{d_{BM}(K,L) - \gamma_m(L)} \frac{m}{1-\gamma_m(L)} = \frac{d_{BM}(K,L)(1-\gamma_m(L))}{d_{BM}(K,L) - \gamma_m(L)} f_m(L), \end{aligned}$$

with equality if and only if $\gamma_m(K) = \frac{\gamma_m(L)}{d_{BM}(K,L)}$.

Since $\gamma_m(L) \le 1/2$,

$$\frac{1-\gamma_m(L)}{d_{BM}(K,L) - \gamma_m(L)} \ge \frac{1}{2d_{BM}(K,L) - 1},$$

with equality if and only if either $d_{BM}(K,L) = 1$ or $d_{BM}(K,L) > 1$ with $\gamma_m(L) = 1/2$. Thus (4) is satisfied and equality holds if and only if either $d_{BM}(K,L) = 1$ or $d_{BM}(K,L) > 1$ with $\gamma_m(K) = \frac{\gamma_m(L)}{d_{BM}(K,L)} = \frac{1}{2d_{BM}(K,L)}$.

3.3 Covering numbers

For a d-dimensional convex body K, we denote by $N_\lambda(K)$ the minimum number of homothetic copies of K of homothety ratio $0 < \lambda \leq 1$ needed to cover K. It follows that $N_\lambda(K) = N(K, \lambda K)$, where $N(K,L)$ is the classical covering number defined as the number of translates of a convex body $L \in \mathscr{K}^d$ needed to cover a convex body $K \in \mathscr{K}^d$. As seen in Section 1, if $K = B^d$, we write $N_{d,\lambda}$ instead of $N_\lambda(B^d)$. Clearly, $N_1(K) = 1$,

$$N_{\gamma_m(K)}(K) \leq m \tag{11}$$

and

$$\gamma_{N_\lambda(K)}(K) \leq \lambda. \tag{12}$$

Moreover, either inequality can be strict. To see that (11) can be strict, consider the example of an affine regular convex hexagon H. Lassak [18] proved that $\gamma_7(K) = 1/2$ holds for any o-symmetric planar convex body K. Thus $\gamma_7(H) = 1/2$. On the other hand, from Figure 1 and the monotonicity of $\gamma_m(K)$ in m [32] it follows that $1/2 = \gamma_7(H) \leq \gamma_6(H) \leq 1/2$. Thus $\gamma_6(H) = 1/2$ and $N_{\gamma_7(H)} = N_{1/2}(H) \leq 6$.

To see that (12) can be strict, note that it is possible to have $N_{\lambda_1}(K) = N_{\lambda_2}(K)$, for some $\lambda_1 < \lambda_2$. For instance, $N_{1/2}(C^d) = N_\lambda(C^d) = 2^d$, for any $1/2 < \lambda < 1$. Therefore, $\gamma_{N_\lambda(C^d)}(C^d) = \gamma_{2^d}(C^d) = 1/2 < \lambda$, for any $1/2 < \lambda < 1$. We use these ideas in the remainder of this section.

3.4 The covering index of direct vector sums of convex bodies

Definition 2 We say that a convex body $K \in \mathcal{K}^d$ is *tightly covered* if for any $0 < \lambda < 1$, K contains $N_\lambda(K)$ points no two of which belong to the same homothet of K with homothety ratio λ.

For instance, $\ell \in \mathcal{K}^1$ is tightly covered since for any $0 < \lambda < 1$, the line segment ℓ contains $N_\lambda(\ell) = \lceil \lambda^{-1} \rceil$ points, no two of which can be covered by the same homothet of the form $\lambda\ell + t$, $t \in \mathbb{E}^1$. Later we will see that for any $d \geq 2$, the d-dimensional cube C^d is also tightly covered. Furthermore, not all convex bodies are tightly covered as will be seen through the example of the circle B^2.

Theorem 2 *Let $\mathbb{E}^d = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n$ be a decomposition of $\mathbb{E}^d$ into the direct vector sum of its linear subspaces $\mathbb{L}_i$ and let $K_i \subseteq \mathbb{L}_i$ be convex bodies such that $\operatorname{coin}(K_i) = f_{m_i}(K_i)$, $i = 1, \ldots, n$, and $\Gamma = \max\{\gamma_{m_i}(K_i) : 1 \leq i \leq n\}$. If some $n-1$ of the K_i's are tightly covered, then*

$$\begin{aligned}
&\max\{\operatorname{coin}(K_i) : 1 \leq i \leq n\} \leq \\
&\qquad \operatorname{coin}(K_1 \oplus \cdots \oplus K_n) = \inf_{\lambda \leq \frac{1}{2}} \frac{\prod_{i=1}^n N_\lambda(K_i)}{1-\lambda} \\
&\qquad\qquad \leq \frac{\prod_{i=1}^n N_\Gamma(K_i)}{1-\Gamma} \leq \frac{\prod_{i=1}^n m_i}{1-\Gamma} < \prod_{i=1}^n \operatorname{coin}(K_i),
\end{aligned} \tag{13}$$

where $K_1 \oplus \cdots \oplus K_n$ stands for the direct sum of the convex bodies $K_1 \subseteq \mathbb{L}_1, \ldots, K_n \subseteq \mathbb{L}_n$. Moreover, the first two upper bounds in (13) are tight.

The core part of the proof of Theorem 2

Proposition 5 *If some $n-1$ of the K_i's are tightly covered, then for all $0<\lambda<1$,*

$$N_\lambda(K_1 \oplus \cdots \oplus K_n) = \prod_{i=1}^{n} N_\lambda(K_i). \tag{14}$$

Hence, for any $0<\lambda<1$,

$$\frac{N_\lambda(K_1 \oplus \cdots \oplus K_n)}{1-\lambda} = \frac{\prod_{i=1}^{n} N_\lambda(K_i)}{1-\lambda}.$$

Thus,

$$\begin{aligned}
\mathrm{coin}(K_1 \oplus \cdots \oplus K_n) &= \inf_{m\in\mathbb{N}} \left\{ \frac{m}{1-\gamma_m(K_1 \oplus \cdots \oplus K_n)} : \gamma_m(K_1 \oplus \cdots \oplus K_n) \le \frac{1}{2}, \right\} \\
&= \inf_{\lambda \le \frac{1}{2}} \frac{N_\lambda(K_1 \oplus \cdots \oplus K_n)}{1-\lambda} \\
&= \inf_{\lambda \le \frac{1}{2}} \frac{\prod_{i=1}^{n} N_\lambda(K_i)}{1-\lambda},
\end{aligned}$$

completing the proof of the equality appearing in (13).

The upper bounds in (13) now follow from the definition of Γ and m_i, $i=1,\ldots,n$. Moreover, the example of d-cubes, considered as direct vector sums of d 1-dimensional line segments, shows that the first two upper bounds in (13) are tight (cf. Theorem 4). □

Corollaries of (the proof of) Theorem 2

Corollary 1 *Let $\mathbb{E}^d = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n$ be a decomposition of $\mathbb{E}^d$ into the direct vector sum of its linear subspaces $\mathbb{L}_i$ and let $K_i \subseteq \mathbb{L}_i$, $i = 1, \ldots, n$, be tightly covered convex bodies. Then $K_1 \oplus \cdots \oplus K_n$ is tightly covered.*

Corollary 2 *Let $\mathbb{E}^d = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_n$ be a decomposition of $\mathbb{E}^d$ into the direct vector sum of its linear subspaces $\mathbb{L}_i$ and let $K_i \subseteq \mathbb{L}_i$ be convex bodies such that $\mathrm{coin}(K_i) = f_{m_i}(K_i)$, $i = 1, \ldots, n$, and $\Gamma = \max\{\gamma_{m_i}(K_i) : 1 \le i \le n\}$. Then*

$$\begin{aligned} \max\{\mathrm{coin}(K_i) : 1 \le i \le n\} &\le \\ \mathrm{coin}(K_1 \oplus \cdots \oplus K_n) &\le \inf_{\lambda \le \frac{1}{2}} \frac{\prod_{i=1}^{n} N_\lambda(K_i)}{1-\lambda} \\ &\le \frac{\prod_{i=1}^{n} N_\Gamma(K_i)}{1-\Gamma} \le \frac{\prod_{i=1}^{n} m_i}{1-\Gamma} < \prod_{i=1}^{n} \mathrm{coin}(K_i). \end{aligned} \tag{15}$$

Moreover, the first three upper bounds in (15) are tight.

Corollary 3 *For any 1-codimensional $(d+1)$-dimensional cylinder $K \oplus \ell$, the first two upper bounds in (13) become equalities and*

$$\mathrm{coin}(K \oplus \ell) = 4N_{1/2}(K).$$

3.5 The covering index of vector sums of convex bodies

Theorem 3 *Let the convex body K be the vector sum of the convex bodies $K_1, \ldots, K_n$ in $\mathbb{E}^d$, i.e., let $K = K_1 + \cdots + K_n$ such that* $\mathrm{coin}(K_i) = f_{m_i}(K_i)$, *$i = 1, \ldots, n$, and $\Gamma = \max\{\gamma_{m_i}(K_i) : 1 \le i \le n\}$. Then*

$$\mathrm{coin}(K) \le \inf_{\lambda \le \frac{1}{2}} \frac{\prod_{i=1}^{n} N_\lambda(K_i)}{1-\lambda} \le \frac{\prod_{i=1}^{n} N_\Gamma(K_i)}{1-\Gamma} \le \frac{\prod_{i=1}^{n} m_i}{1-\Gamma} < \prod_{i=1}^{n} \mathrm{coin}(K_i). \tag{16}$$

Moreover, equality in (16) does not hold in general.

Corollary 4 *If K is any d-dimensional convex body, such that* $\mathrm{coin}(K) = f_m(K)$. *Then*

$$\mathrm{coin}(K-K) \le \frac{\left(N_{\gamma_m(K)}(K)\right)^2}{1-\gamma_m(K)} \le \frac{m^2}{1-\gamma_m(K)} < (\mathrm{coin}(K))^2. \tag{17}$$

Moreover, equality in (17) does not hold in general.

Problem 2 Let $K_1, \ldots, K_n$ be d-dimensional convex bodies, for some $d \ge 2$. Then prove (disprove) that

$$\max\{\mathrm{coin}(K_i) : i = 1, \ldots, n\} \le \mathrm{coin}(K_1 + \cdots + K_n). \tag{18}$$

If this does not hold, one can try proving the following weaker lower bound.

$$\min\{\mathrm{coin}(K_i) : i = 1, \ldots, n\} \le \mathrm{coin}(K_1 + \cdots + K_n). \tag{19}$$

4. Extremal bodies
4.1 Minimizer in d-space

Theorem 4 *Let d be any positive integer and $K \in \mathscr{K}^d$. Then* $\operatorname{coin}(C^d) = 2^{d+1} \leq \operatorname{coin}(K)$ *and thus (affine) d-cubes minimize the covering index in all dimensions.*

Proof Clearly, C^d can be covered by 2^d homothets of homothety ratio $1/2$, and cannot be covered by fewer homothets. Therefore, $\operatorname{coin}(C^d) \leq f_{2^d}(C^d) = 2^{d+1}$. Let p be a positive integer. If there exists a homothetic covering of C^d by $m = 2^d + p$ homothets giving $f_m(C^d) < 2^{d+1}$, then

$$\gamma_m(C^d) < \frac{1}{2} - \frac{p}{2^{d+1}}.$$

However,

$$m \operatorname{vol}(\gamma_m(C^d)C^d) = m\gamma_m(C^d)^d \operatorname{vol}(C^d) < (2^d + p)\left[\frac{1}{2} - \frac{p}{2^{d+1}}\right]^d \operatorname{vol}(C^d) < \operatorname{vol}(C^d),$$

a contradiction, showing that $\operatorname{coin}(C^d) = 2^{d+1}$.

Now consider an arbitrary d-dimensional convex body K. By repeating the above calculations for K we see that for $m > 2^d$, $f_m(K)$ cannot be smaller than 2^{d+1}. A similar volumetric argument shows that K cannot be covered by 2^d homothets having homothety ratio less than $1/2$. Likewise, it is impossible to cover K by fewer than 2^d homothets if the homothety ratio does not exceed $1/2$. Thus $\operatorname{coin}(K) \geq 2^{d+1}$. □

4.2 Maximizer in the plane

The case of coin-maximizers is more involved. Indeed since we have not established the upper semicontinuity of coin, it may be the case that for some d, $\sup\left\{\mathrm{coin}(K) : K \in \mathscr{K}^d\right\}$ is not achieved by any d-dimensional convex body. However, this is not the case for $d = 2$.

Theorem 5 *If K is a planar convex body then* $\mathrm{coin}(K) \leq \mathrm{coin}(B^2) = 14$.

Proof First, we show that $\mathrm{coin}(B^2) = 14$. It is rather trivial that $\gamma_1(B^2) = \gamma_2(B^2) = 1$, $\gamma_3(B^2) = \sqrt{3}/2 = 0.866\ldots$, and $\gamma_4(B^2) = 1/\sqrt{2} = 0.707\ldots$. Hence, $f_1(B^2) = f_2(B^2) = f_3(B^2) = f_4(B^2) = +\infty$. Moreover, the first named author [2] showed that $\gamma_5(B^2) = 0.609\ldots$ and $\gamma_6(B^2) = 0.555\ldots$, implying that $f_5(B^2) = f_6(B^2) = +\infty$. On the other hand, it is easy to see that $\gamma_7(B^2) = 1/2$ and therefore $f_7(B^2) = 14$. Hence Lemma 1 implies that $\mathrm{coin}(B^2) = \min\left\{f_m(B^2) : 7 \leq m < 14\right\}$.

Next, recall G. Fejes Tóth's result [11] according to which $\gamma_8(B^2) = 0.445\ldots$ and $\gamma_9(B^2) = 1/(1+\sqrt{2}) = 0.414\ldots$. This implies $f_8(B^2) = 14.420\ldots > 14$ and $f_9(B^2) = 15.363\ldots > 14$.

We claim that $f_m(B^2) > 14$, for all $10 \leq m < 14$. Suppose for some $10 \leq m \leq 14$, $f_m(B^2) \leq 14$. In this case, we must have $\gamma_m(B^2) \leq \frac{14-m}{14}$ and $m\,\mathrm{vol}(\gamma_m(B^2)B^2) > \mathrm{vol}(B^2)$. This implies $m\left(\frac{14-m}{14}\right)^2 > 1$. But, routine calculations show that the latter inequality fails to hold for all $10 \leq m \leq 13$. Thus $\mathrm{coin}(B^2) = 14$.

Levi [19] showed that any planar convex body K can be covered by 7 homothets of homothety ratio $1/2$. Thus $\mathrm{coin}(K) \leq 14$, proving that circle maximizes the covering index in the plane. □

The problem of maximizer in d-space for d>2

Problem 3 For any d-dimensional convex body K, prove or disprove that $\mathrm{coin}(K) \leq \mathrm{coin}(B^d)$ holds.

An affirmative answer to Problem 3 would considerably improve the known general (Rogers-type) upper bound on the illumination number. It is known (e.g., see [6]) that for any d-dimensional convex body K, in general

$$I(K) = \binom{2d}{d} d(\ln d + \ln\ln d + 5) = O(4^d d \ln d), \tag{20}$$

and if, in addition, K is o-symmetric, then

$$I(K) = 2^d d(\ln d + \ln\ln d + 5) = O(2^d d \ln d). \tag{21}$$

If B^d maximizes the covering index, then the general bound (20) would improve to $I(K) = O(2^d d^{3/2} \ln d)$ which is within a factor $\sqrt{d}$ of the bound (4) in the o-symmetric case.