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Recent developments in the reduction approach to action-angle dualities of integrable many-body systems

• Integrable systems of Toda and Calogero (Sutherland, Moser, Olshanetsky-Perelomov, Ruijsenaars-Schneider) type describe point "particles" moving on the line or on the circle.

• These important systems enjoy intriguing "duality relations".

• By definition, two integrable many-body systems are dual to each other if the action variables of system (i) are the particle positions of system (ii), and vice versa.

• A special case of duality is self-duality, where the leading Hamiltonians of the two systems have the same form.

Elaboration of the definition of duality

Consider "Liouville integrable" Hamiltonian systems (M, ω, H) and $(\widetilde{M}, \tilde{\omega}, \widetilde{H})$. These systems are said to be **in action-angle duality** if there exist Darboux coordinates q_i, p_i on (dense open subset of) M and Darboux coordinates $\hat{p}_k, \hat{q}_k$ on (dense open subset of) $\widetilde{M}$ and a global symplectomorphism $\mathcal{R} : M \to \widetilde{M}$ such that

$H \circ \mathcal{R}^{-1}$ depends only on $\hat{p}$ (action variables for H) and

$\widetilde{H} \circ \mathcal{R}$ depends only on q (action variables for $\widetilde{H}$).

"This is non-trivial if the two systems are of interest."

It is a particularly interesting relation if both are many-body systems (interacting points) in such a way that

the q_i describe particle positions for $H(q,p)$ and

the $\hat{p}_i$ describe particle positions for $\widetilde{H}(\hat{p}, \hat{q})$.

• Equivalent definition of self-duality: an integrable many-body Hamiltonian system is called self-dual if it admits a global symplectomorphism that "exchanges" its particle position and action variables. (The "self-duality map" usually has order 4.)

• First example is the self-duality of the rational Calogero system. Interpreted in terms of symplectic reduction by Kazhdan, Kostant and Sternberg (1978).

• Duality was discovered and explored by Ruijsenaars (1988-95) in his direct construction of action-angle variables for Calogero and Toda type systems and their relativistic deformations.

• The duality map is the same as the "action-angle" map, thus one may use the term "action-angle duality" (also fitting to call it "Ruijsenaars duality").

• This duality is classical analogue of (quantum . . .) bispectrality.

Well-known examples

Rational Calogero system is self-dual

$$H_{\mathrm{Cal}}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{x^2}{(q_k - q_j)^2}$$

Hyperbolic Sutherland system

$$H_{\mathrm{hyp-Suth}}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{x^2}{\sinh^2(q_j - q_k)}$$

is in duality with rational Ruijsenaars-Schneider system

$$H_{\mathrm{rat-RS}}(\hat{p},\hat{q}) = \sum_{k=1}^{n} \cosh(\hat{q}_k) \prod_{j\neq k} \left[1 + \frac{x^2}{(\hat{p}^k - \hat{p}^j)^2}\right]^{\frac{1}{2}}$$

Treated via Hamiltonian reduction in arXiv:0901.1983 with Klimcik.

Duality from Hamiltonian reduction: the basic idea

Start with 'big phase space', of group theoretic origin, equipped with *two* commuting families of 'canonical free Hamiltonians'.

Apply suitable reduction to the big phase space and construct *two* 'natural' models of the *single* reduced phase space.

The two families of 'free' Hamiltonians turn into commuting **many-body Hamiltonians** (or 'actions') and **particle positions** in terms of *both* models. Their rôle is *interchanged* in the two models.
In practice (roughly), big phase space consists of pairs of matrices (A, B) and the two 'models' are two 'gauge slices' defined by 'diagonalizing' A or B. Particle positions and actions descend from eigenvalues (or other invariants) of A and B.

The natural symplectomorphism between the two models of the reduced phase space yields the 'duality symplectomorphism'.

• My purpose (work of last 6 years) is to derive all known dualities by reductions of suitable (finite-dimensional, real) phase spaces and to find new dual pairs.

• Among others, we proved conjectures and "globalized" local results of Gorsky-Nekrasov [95] and Fock-Gorsky-Nekrasov-Roubtsov [2000].

Further examples of dual pairs

Trigonometric Sutherland system and its Ruijsenaars dual

$$H_{\text{trigo}-\text{Suth}} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{x^2}{2}\sum_{j\neq k}\frac{1}{\sin^2(q_k - q_j)}$$

$$\widetilde{H}_{\text{rat}-\text{RS}} = \sum_{k=1}^{n}(\cos\hat{q}_k)\prod_{j\neq k}\left[1 - \frac{x^2}{(\hat{p}_k - \hat{p}_j)^2}\right]^{\frac{1}{2}}$$

Described in terms of reduction of $T^*U(n)$ (Ayadi-Feher 2010).

'Relativistic' deformation of above dual pair:

$$H_{\text{trigo}-\text{RS}} = \sum_{k=1}^{n}(\cosh p_k)\prod_{j\neq k}\left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_j)}\right]^{\frac{1}{2}}$$

and the **physically very different** dual system

$$\widehat{H}_{\text{trigo}-\text{RS}} = \sum_{k=1}^{n}(\cos\hat{q}_k)\prod_{j\neq k}\left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_k - \hat{p}_j)}\right]^{\frac{1}{2}}$$

Derived by reduction of Heisenberg double of P-L $U(n)$ (Feher-Klimcik 2009).

Plan of what follows

First, flash **new** cases of duality associated with BC_n root system.
(Detailed treatment in recent joint paper with T.F. Gorbe, arXiv:1407.2057)

Second, (following Feher-Kluck: arXiv:1312.0400) report **new** compact self-dual systems, locally given by Ruijsenaars' "III_b Hamiltonian"

$$H_{\text{compact}-\text{RS}} = \sum_{k=1}^{n} (\cos p_k) \prod_{j\neq k} \left| 1 - \frac{\sin^2 x}{\sin^2(q_k - q_j)} \right|^{\frac{1}{2}} \quad \textbf{for generic } 0 < x < \pi$$

Earlier with Klimcik, arXiv:1101.1759, studied the "standard case" $0 < x < \pi/n$.

Third, describe novel group-theoretic interpretation of old results about action-angle map and duality for open Toda

$$H_{\text{Toda}}(q,p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} \quad \text{(based on arXiv:1312.0404)}$$

The third topic is related to works by Ruijsenaars (1990), Babelon (2003), Sklyanin (2013), Kozlowski (2013), and others.

New dual pairs associated with BC_n root system

Trigonometric BC_n Sutherland system

$$H(q,p) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{1\leq j<k\leq n}\left(\frac{\gamma}{\sin^2(q_j-q_k)}+\frac{\gamma}{\sin^2(q_j+q_k)}\right)+\sum_{j=1}^{n}\frac{\gamma_1}{\sin^2(q_j)}+\sum_{j=1}^{n}\frac{\gamma_2}{\sin^2(2q_j)}$$

is dual to (**completion of**) rational Ruijsenaars-Schneider-van Diejen system

$$\widetilde{H}(\lambda,\vartheta) = \sum_{j=1}^{n}\cos(\vartheta_j)\left[1-\frac{\nu^2}{\lambda_j^2}\right]^{\frac{1}{2}}\left[1-\frac{\kappa^2}{\lambda_j^2}\right]^{\frac{1}{2}}\prod_{\substack{k=1\\(k\neq j)}}^{n}\left[1-\frac{4\mu^2}{(\lambda_j-\lambda_k)^2}\right]^{\frac{1}{2}}\left[1-\frac{4\mu^2}{(\lambda_j+\lambda_k)^2}\right]^{\frac{1}{2}}$$

$$-\frac{\nu\kappa}{4\mu^2}\prod_{j=1}^{n}\left[1-\frac{4\mu^2}{\lambda_j^2}\right]+\frac{\nu\kappa}{4\mu^2}.$$

Coupling constants are subject to $\gamma>0$, $\gamma_2>0$, $4\gamma_1+\gamma_2>0$ and $\mu>0$, $\nu>|\kappa|\geq 0$. Duality holds under the relation $\gamma=\mu^2$, $\gamma_1=\frac{\nu\kappa}{2}$, $\gamma_2=\frac{(\nu-\kappa)^2}{2}$.

Sutherland positions q satisfy $\frac{\pi}{2}>q_1>\cdots>q_n>0$ and Sutherland actions λ (dual poisitions) fill closure of domain

$$D=\{\lambda\in\mathbb{R}^n \mid \lambda_a-\lambda_{a+1}>2\mu\ (a=1,\ldots,n-1),\ \lambda_n>\nu\}.$$

Liouville tori collapse at boundary of D, and above description of the dual system is valid on dense open submanifold $D\times\mathbb{T}^n$ of the phase space.

This generalizes earlier result for hyperbolic BC_n Sutherland system obtained by Pusztai, arXiv:1109.0446, using reduction of $T^*U(n,n)$. See arXiv:1407.2057 with T.F. Gorbe, treating suitable Hamiltonian reduction of $T^*U(2n)$.

On new compact forms of the trigonometric RS systems

Consider $G := SU(n)$ and equip the double $G \times G = \{(A,B)\}$ with the 2-form
$\omega^\lambda := \lambda\left(\langle A^{-1}dA \overset{\wedge}{,} dBB^{-1}\rangle + \langle dAA^{-1} \overset{\wedge}{,} B^{-1}dB\rangle - \langle (AB)^{-1}d(AB) \overset{\wedge}{,} (BA)^{-1}d(BA)\rangle\right).$

The 2-form, the moment map $\mu\colon (A,B) \mapsto ABA^{-1}B^{-1}$, and the action of G by componentwise conjugation makes $G \times G$ a quasi-Hamiltonian space (Alekseev-Malkin-Meinrenken, 1998).

The reduced phase space $P(\mu_0) := \mu^{-1}(\mu_0)/G_{\mu_0}$ is symplectic.

The class functions of G, applied to either components A or B in the pair $(A,B) \in G \times G$, descend to two Abelian Poisson algebras on $P(\mu_0)$.

Earlier with Klimcik, analyzed this quasi-Hamiltonian reduction taking

$$\mu_0 := \mu_0(x) := \mathrm{diag}\left(e^{2\mathrm{i}x}, \ldots, e^{2\mathrm{i}x}, e^{-2\mathrm{i}(n-1)x}\right)$$

with $0 < x < \pi/n$. Now with Timo Kluck, studied the case of generic $0 < x < \pi$.

First result: this construction always gives a self-dual integrable system on the compact, connected, smooth reduced phase space $P(\mu_0(x))$ of dimension $2(n-1)$.

Second result: On a dense open submanifold of $P(\mu_0(x))$ the "main Hamiltonian" coming from $\Re\,(\mathrm{tr}\,(A))$ takes the RS form of $\mathrm{III_b}$ type:

$$H_{\text{compact}-\text{RS}} = \sum_{k=1}^{n} (\cos p_k) \prod_{j\neq k} \left| 1 - \frac{\sin^2 x}{\sin^2(q_k - q_j)} \right|^{\frac{1}{2}}$$

This describes n "particles" moving on the circle. Domain of "position variables" is the same as domain of "action variables" and depends on value of x.

Two types of compact RS systems

The analysis requires finding the spectra of B for all (A, B) in the constraint surface $\mu^{-1}(\mu_0(x))$, where $ABA^{-1}B^{-1} = \mu_0(x)$. $/e^{2\mathrm{i}xm} \neq 1$ for all $m = 1, 2, \ldots, n/$

In principle, two qualitatively different types of cases can occur:

- Type (i): the constraint surface satisfies $\mu^{-1}(\mu_0(x)) \subset G_{\text{reg}} \times G_{\text{reg}}$.
- Type (ii): the relation $\mu^{-1}(\mu_0(x)) \subset G_{\text{reg}} \times G_{\text{reg}}$ does not hold.

The reduced phase space $P(\mu_0(x))$ is naturally a Hamiltonian toric manifold if and only if $\mu^{-1}(\mu_0(x)) \subset G_{\text{reg}} \times G_{\text{reg}}$, i.e., in the type (i) cases. In other words, one obtains $(n-1)$ globally smooth, independent action variables generating an effective torus action.

Indeed, in the type (i) cases certain "spectral functions" on G that are smooth on G_{reg} but only continuous at G_{sing} descend to smooth action variables and position variables when applied to A and B with $(A, B) \in \mu^{-1}(\mu_0(x))$.

In the type (ii) cases the particles can collide and the action variables become non-differentiable at singular points, where the $(n-1)$ commuting smooth Hamiltonians loose their independence.

Our main result: We found the complete classification of the parameter $0 < x < \pi$ according to type (i) and type (ii) cases.

Classification of the coupling parameter

Main Theorem: The type (i) cases are precisely those for which the coupling parameter $0 < x < \pi$ (subject to $e^{2\mathrm{i}xm} \neq 1$ for all $m = 1, 2, \ldots, n$) belongs to an open interval of the form

$$\pi \left(\frac{c}{n} - \frac{1}{nd}, \frac{c}{n} + \frac{1}{(n-d)n} \right)$$

with integers c, d satisfying $1 \leq c, d \leq (n-1)$, $\gcd(n, c) = 1$ and $cd = 1 \mod n$. In these cases the reduced phase space $P(\mu_0(x))$ is symplectomorphic to $\mathbb{C}P^{n-1}$ endowed with a multiple of the Fubini-Study symplectic structure.

- The result was obtained by determining the possible spectra of the matrix B satisfying $ABA^{-1}B^{-1} = \mu_0(x)$.
- In the type (i) cases we found that the "Delzant polytope" is a simplex.
- The existence of type (ii) cases is completely new.
- The only previously studied type (i) cases are those for which $c = 1$ and x belongs to the "half-interval" $(0, \pi/n)$.
- The detailed description of our new integrable systems is still largely open.

Turning to our third topic, recall that Ruijsenaars (1990) found explicit action-angle map for open Toda and introduced dual integrable system.

$$M := \mathbb{R}^n \times \mathbb{R}^n = \{(q,p)\}, \quad \omega = \sum_{i=1}^{n} dp_i \wedge dq_i, \quad H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}$$

Phase space of action-angle variables: $(\hat{M}, \hat{\omega})$

$$\hat{M} := \{(\hat{p}, \hat{q}) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, \hat{p}_1 > \hat{p}_2 > \cdots > \hat{p}_n\}, \qquad \hat{\omega} = \sum_{i=1}^{n} d\hat{q}_i \wedge d\hat{p}_i$$

Formula of action-angle map $R : \hat{M} \to M$

$$q_j = \ln(\sigma_{n+1-j}/\sigma_{n-j}), \qquad p_j = \dot{\sigma}_{n+1-j}/\sigma_{n+1-j} - \dot{\sigma}_{n-j}/\sigma_{n-j},$$

$$\sigma_k := \sum_{|I|=k} e^{\sum_{l\in I}\hat{q}_l} \prod_{i\in I, j\notin I} |\hat{p}_i - \hat{p}_j|^{-1} \qquad (\forall k = 1, \ldots, n, \quad \sigma_0 := 1)$$

$I \subset \{1, 2, \ldots, n\}$ subset of cardinality $|I| = k$, $\dot{\sigma}_k := \{\sigma_k, \frac{1}{2}\sum_{i=1}^n \hat{p}_i^2\}_{\hat{M}}$

Action-angle map R converts H into free form: $H \circ R = \frac{1}{2}\sum_{i=1}^n \hat{p}_i^2$.

Dual system: $(\hat{M}, \hat{\omega}, \hat{H})$ with $\hat{H} := \sigma_1 = e^{q_n} \circ R = \sum_{i=1}^n e^{\hat{q}_i} \prod_{j\neq i} \frac{1}{|\hat{p}_i - \hat{p}_j|}$

Toda action-angle map and duality from symplectic reduction

Unreduced phase space: $T^*GL(n,\mathbb{R}) \simeq GL(n,\mathbb{R}) \times gl(n,\mathbb{R}) = \{(g,\mathcal{J})\}$ equipped with symplectic form $\Omega := 2d\mathrm{tr}\,(\mathcal{J}g^{-1}dg)$.

Two sets of commuting "free Hamiltonians" $\{\mathcal{H}_k\}$ and $\{\widehat{\mathcal{H}}_k\}$:

$$\mathcal{H}_k(g,\mathcal{J}) := \frac{1}{k}\mathrm{tr}\,(\mathcal{J}^k), \qquad \widehat{\mathcal{H}}_k(g,\mathcal{J}) := m_k((gg^t)^{-1}), \quad k = 1,\ldots,n,$$

Notation: $m_k(X) := \det(X_k)$ is k-th leading principal minor of $n\times n$ matrix X.

Reduce by the symmetry group $N_+ \times O(n,\mathbb{R})$. N_+ is upper triangular nilpotent subgroup and (η_+,η_O) from symmetry group acts by the map $\Psi_{(\eta_+,\eta_O)}$:

$$\Psi_{(\eta_+,\eta_O)}(g,\mathcal{J}) := (\eta_+ g\eta_O^{-1}, \eta_O\mathcal{J}\eta_O^{-1}).$$

This Hamiltonian action is generated by the moment map Φ:

$$\Phi(g,\mathcal{J}) = ((g\mathcal{J}g^{-1})_{\text{lower–triangular part}}, -\mathcal{J}_{\text{anti–symmetric part}}).$$

Reduction relevant for Toda is defined by imposing the moment map constraint

$$\Phi(g,\mathcal{J}) = \mu_0 := (I_-, 0), \qquad \text{(Olshanetsky-Perelomov, Adler, Kostant, Symes, \ldots)}$$

where $I_- := \sum_{i=1}^{n-1} E_{i+1,i}$ contains 1 in its entries just below the diagonal.

Reduced phase space $\Phi^{-1}(\mu_0)/(N_+ \times O(n,\mathbb{R}))$ inherits 2 Abelian Poisson algebras.

First model of the reduced phase space: 'Toda gauge'

By Iwasawa decomposition, any $g \in GL(n,\mathbb{R})$ can be uniquely written as

$$g = g_+ g_A g_O, \qquad (g_+, g_A, g_O) \in N_+ \times A \times O(n,\mathbb{R}).$$

Associate to $(q,p) \in M := \mathbb{R}^n \times \mathbb{R}^n$ the diagonal matrices

$$Q(q) := -\sum_{i=1}^{n} q_{n+1-i} E_{i,i}, \quad P(p) := -\sum_{i=1}^{n} p_{n+1-i} E_{i,i}$$

and define Jacobi matrix (alias Toda Lax matrix, since $H(q,p) = \frac{1}{2}\mathrm{tr}\,(L(q,p)^2)$)

$$L(q,p) := P(p) + e^{-Q(q)/2} I_- e^{Q(q)/2} + e^{Q(q)/2} I_+ e^{-Q(q)/2}.$$

The following manifold S is a global cross section of the orbits of the "gauge group" $N_+ \times O(n,\mathbb{R})$ in the "constraint surface" $\Phi^{-1}(\mu_0)$:

$$S := \{(e^{Q(q)/2}, L(q,p)) \,|\, (q,p) \in M\}.$$

Reduced symplectic form is represented by pull-back $\iota_S^*(\Omega) = \sum_{i=1}^n dp_i \wedge dq_i \equiv \omega$.

The equalities $\quad \iota_S^*(\mathcal{H}_k) = \dfrac{1}{k}\mathrm{tr}\,(L^k), \quad \iota_S^*(\widehat{\mathcal{H}}_k) = \displaystyle\prod_{j=1}^{k} e^{q_{n+1-j}} \quad$ show that

in terms of model $(S, \iota_S^*(\Omega)) \simeq (M, \omega)$ of reduced phase space, the unreduced free Hamiltonians $\{\mathcal{H}_k\}$ descend to commuting Toda Hamiltonians and $\{\widehat{\mathcal{H}}_k\}$ descend to (functions of) Toda position variables.

All this is well-known. I call S 'Toda gauge': a model of $\Phi^{-1}(\mu_0)/(N_+ \times O(n,\mathbb{R}))$.

Second model of the reduced phase space: 'Moser gauge'

$\mathbb{R}^n_>$: set of vectors $\hat{p}$ satisfying $\hat{p}_1 > \hat{p}_2 > \cdots > \hat{p}_n$. $\mathbb{R}^n_+$: vectors w having positive components. For $(\hat{p}, w) \in \mathbb{R}^n_> \times \mathbb{R}^n_+$ define $n \times n$ matrices Λ and Γ by

$$\Lambda(\hat{p}) := \mathrm{diag}(\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n), \qquad \Gamma(\hat{p}, w)_{i,k} := w_i \, (\hat{p}_i)^{k-1} \quad (\texttt{diagonal} \times \texttt{Vandermonde})$$

My main observation: *The manifold*

$$\hat{S} := \{(\Gamma(\hat{p}, w)^{-1}, \Lambda(\hat{p})) \,|\, (\hat{p}, w) \in \mathbb{R}^n_> \times \mathbb{R}^n_+\}$$

is a global cross-section of the orbits of $N_+ \times O(n, \mathbb{R})$ in constraint surface $\Phi^{-1}(\mu_0)$.

The key is to consider Iwasawa decomposition

$$\Gamma(\hat{p}, w)^{-1} = \eta_+(\hat{p}, w) \rho(\hat{p}, w) \eta_O(\hat{p}, w) \text{ with } \rho(\hat{p}, w) = \mathrm{diag}(\rho_1(\hat{p}, w), \dots, \rho_n(\hat{p}, w)).$$

Fact: $\eta_O(\hat{p}, w) \Lambda(\hat{p}) \eta_O(\hat{p}, w)^{-1}$ is Jacobi matrix, determines $(\hat{p}, w)$ up to scale of w.

Then unique gauge transformation from $\hat{S}$ to S yields a map

$$\mathcal{R} : \hat{S} \to S, \qquad (\hat{p}, w) \mapsto (e^{Q(q)/2}, L(q, p)) = (\rho(\hat{p}, w), \eta_O(\hat{p}, w) \Lambda(\hat{p}) \eta_O(\hat{p}, w)^{-1}).$$

It is EASY to find this map explicitly since $\Gamma(\hat{p}, w)$ is $\texttt{diagonal} \times \texttt{Vandermonde}$.

Using Cauchy-Binet, trivial calculation of $\hat{\mathcal{H}}_k(g, \mathcal{J}) = m_k((gg^t)^{-1})$ in the two gauges gives

$$\prod_{j=1}^{k} e^{q_{n+1-j}} \circ \mathcal{R} = m_k(\Gamma(\hat{p}, w)^t \Gamma(\hat{p}, w)) = \sum_{|I|=k} \Bigg(\prod_{l \in I} w_l^2 \prod_{\substack{i,j \in I \\ i \neq j}} |\hat{p}_i - \hat{p}_j| \Bigg).$$

To finish, parametrize Moser's variables $(\hat{p}, w)$ by Darboux coordinates $(\hat{p}, \hat{q})$.

Ruijsenaars' action-angle map and duality from reduction

Reduced symplectic form is easily calculated in the Moser gauge

$$\iota_{\widehat{S}}^*(\Omega) = 2\sum_{i=1}^n d\ln w_i \wedge d\widehat{p}_i + \sum_{\substack{j,k=1\\ j\neq k}}^n \frac{d\widehat{p}_j \wedge d\widehat{p}_k}{\widehat{p}_j - \widehat{p}_k}$$

(thanks to C. Klimcik)

Corresponding Poisson brackets: $\{\widehat{p}_i, \widehat{p}_j\} = 0$, $\{\widehat{p}_i, w_j\} = \frac{w_j}{2}\delta_{ij}$, $\{w_j, w_k\} = \frac{1}{2}\frac{w_j w_k}{\widehat{p}_j - \widehat{p}_k}$.
These variables linearize the Toda flows, whose Hamiltonians become on $\widehat{S}$

$$\iota_{\widehat{S}}^*(\mathcal{H}_k) = \frac{1}{2}\sum_{i=1}^n (\widehat{p}_i)^k.$$

Toda action-angle variables $(\widehat{p}, \widehat{q})$ are obtained by the parametrization

$$w_i(\widehat{p}, \widehat{q}) := e^{\frac{1}{2}\widehat{q}_i} \prod_{\substack{j=1\\ j\neq i}}^n |\widehat{p}_i - \widehat{p}_j|^{-\frac{1}{2}}, \qquad (\widehat{p}, \widehat{q}) \in \mathbb{R}^n_> \times \mathbb{R}^n \equiv \widehat{M},$$

which brings $\iota_{\widehat{S}}^*(\Omega)$ into Darboux form $\widehat{\omega} = \sum_{i=1}^n d\widehat{q}_i \wedge d\widehat{p}_i$.

Map $\mathcal{R} : \widehat{S} \to S$ is automatically symplectomorphism, and "explains" Ruijsenaars' formula. Reduced Hamiltonians $\iota_{\widehat{S}}^*(\widehat{\mathcal{H}}_k)$ are Ruijsenaars' dual Hamiltonians. Toda position variables q_k are action variables of main dual Hamiltonian:

$$\widehat{H} = e^{q_n} \circ \mathcal{R} = \iota_{\widehat{S}}^*(\widehat{\mathcal{H}}_1) = \sum_{i=1}^n w_i^2 = \sum_{i=1}^n e^{\widehat{q}_i} \prod_{j\neq i} \frac{1}{|\widehat{p}_i - \widehat{p}_j|}.$$

CONCLUSION

Main point of the reduction approach:

Once the correct starting point is ‘guessed’, GLOBAL phase spaces and duality symplectomorphisms result automatically.

This approach links integrable many-body systems and their duality to a host of interesting subjects.

We applied reduction methods to several many-body systems and obtained group-theoretic interpretation of their duality relations.

Major open questions: How to obtain the hyperbolic RS system?

How to deal with relativistic Toda?

All BC_n RS-vD systems from reduction?

(related work by Marshall, arXiv:1311.4641)

What about the elliptic systems?