Towards a logarithmic Brunn-Minkowski theory

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Minkowski problem

- K, C - convex bodies in $\mathbb{R}^n$
- h_k - support function of K
- S_K - surface area measure of K on S^{n-1}
- L - linear subspace, $L \neq \{o\}, \mathbb{R}^n$
- μ - non-trivial Borel measure on S^{n-1}

"Minkowski problem"

$\mu = S_K$ for some unique convex body K (up to translation) iff

1. $\mu(L \cap S^{n-1}) < \mu(S^{n-1})$ for any L with $\dim L = n - 1$
2. $\int_{S^{n-1}} u \, d\mu(u) = o$

If μ is even, then the first condition is enough

Brunn-Minkowski inequality

$\alpha, \beta > 0$
$\alpha K + \beta C = \{x \in \mathbb{R}^n : \langle u, x \rangle \le \alpha h_K(u) + \beta h_C(u) \ \forall u \in S^{n-1}\}$

Brunn-Minkowski inequality $\quad \alpha \in (0,1)$

$$V(\alpha K + (1-\alpha)C) \ge V(K)^\alpha V(C)^{1-\alpha}$$

with equality iff K and C are homothetic.

Minkowski inequality If $V(K) = V(C)$, then

$$\int_{S^{n-1}} h_C \, dS_K \ge \int_{S^{n-1}} h_K \, dS_K,$$

with equality iff K and C are translates.

To solve the Minkowski problem,

- Minimize $\int_{S^{n-1}} h_C \, d\mu$ under the condition $V(C) = 1$
- Uniqueness comes from the Minkowski inequality

Logarithmic Minkowski problem - Cone volume measure

$dV_K = \frac{1}{n}\, h_K\, dS_K$ - cone volume measure on S^{n-1} if $o \in \mathrm{int}\, K$

Theorem (Even logarithmic Minkowski problem, BLYZ)

Let μ be an even Borel measure on S^{n-1}.
$\mu = V_K$ for some o-symmetric convex body K iff

- (i) $\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n}\, \mu(S^{n-1})$ *for any* $L \neq \{o\}, \mathbb{R}^n$
- (ii) *If equality holds for some L, then $\mathrm{supp}\, \mu \subset L \cup L'$ for some complementary L'*

Necessity for polytopes: Henk-Schuermann-Wills, He-Leng, Xiong
Idea for sufficiency: Minimize $\int_{S^{n-1}} \log h_C\, d\mu$ assuming $V(C) = 1$

Conjecture (Uniqueness)

$V_K = V_C$ for o-symmetric convex bodies K and C with $V(K) = V(C)$ iff K and C have dilated direct summands; namely, $K = K_1 \oplus \ldots \oplus K_m$ and $C = C_1 \oplus \ldots \oplus C_m$ with $K_i = \lambda_i C_i$ for $\lambda_1, \ldots, \lambda_m > 0$.

Isotropic position of a measure on S^{n-1}

Theorem (BLYZ)

Let μ be a Borel probability measure on S^{n-1}. There exists $A \in \mathrm{GL}(n)$ such that

$$\int_{S^{n-1}} \frac{Au}{\|Au\|} \otimes \frac{Au}{\|Au\|}\, d\mu(u) = \tfrac{1}{n}\,\mathrm{Id}_n$$

iff

(i) $\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n}$ *for any* $L \neq \{o\}, \mathbb{R}^n$

(ii) *If equality holds for some L, then* $\operatorname{supp}\mu \subset L \cup L'$ *for some complementary L'*

- Sufficiency if $\mu(L \cap S^{n-1}) < \frac{\dim L}{n}$ is due to Klartag (supergaussian marginals of probability measures on $\mathbb{R}^n$)
- Discrete case is due to Carlen-Lieb-Loss (extremals for the Brascamp-Lieb inequality). See also Benneth-Carbery-Christ-Tao, Carlen&Cordero-Erausquin

Logarithmic Brunn-Minkowski inequality

$\alpha \in [0,1]$, $o \in \mathrm{int} K, \mathrm{int} C$

$\alpha K +_0 (1-\alpha) C = \{x \in \mathbb{R}^n : \langle u, x\rangle \leq h_K(u)^\alpha h_C(u)^{1-\alpha} \ \forall u \in S^{n-1}\}$

$\alpha K +_0 (1-\alpha) C \ \subset \ \alpha K + (1-\alpha) C$

Conjecture (Logarithmic Brunn-Minkowski conjecture)

$\alpha \in (0,1)$, K, C are o-symmetric

$$V(\alpha K +_0 (1-\alpha) C) \geq V(K)^\alpha V(C)^{1-\alpha}$$

with equality iff K and C have dilated direct summands.

Conjecture (Logarithmic Minkowski conjecture)

For o-symmetric K, C, if $V(K) = V(C)$, then

$$\int_{S^{n-1}} \log h_C \, dV_K \geq \int_{S^{n-1}} \log h_K \, dV_K,$$

with equality iff K and C have dilated direct summands.

About the logarithmic Brunn-Minkowski conjecture

- Interesting for any log-concave measure instead of volume
- $n = 2$ for volume (BLYZ)
- K and C are unconditional for any log-concave measure (Bollobás&Leader and Cordero-Erausquin&Fradelizi&Maurey on coordinatewise product)
- K and C are dilates for the gaussian measure (Cordero-Erausquin&Fradelizi&Maurey on B-conjecture)