When is a disk trapped by four lines ?

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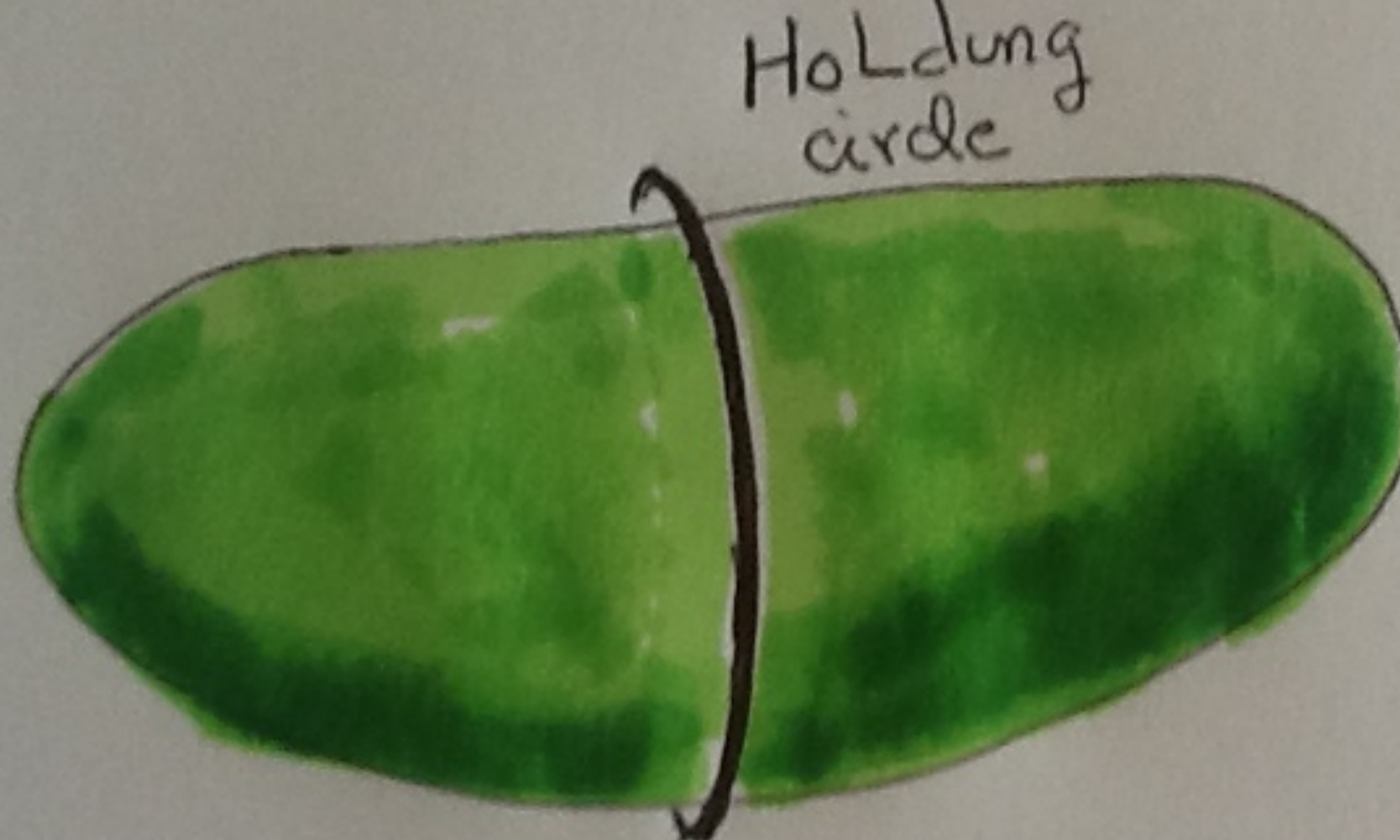
Szeged , September 2013

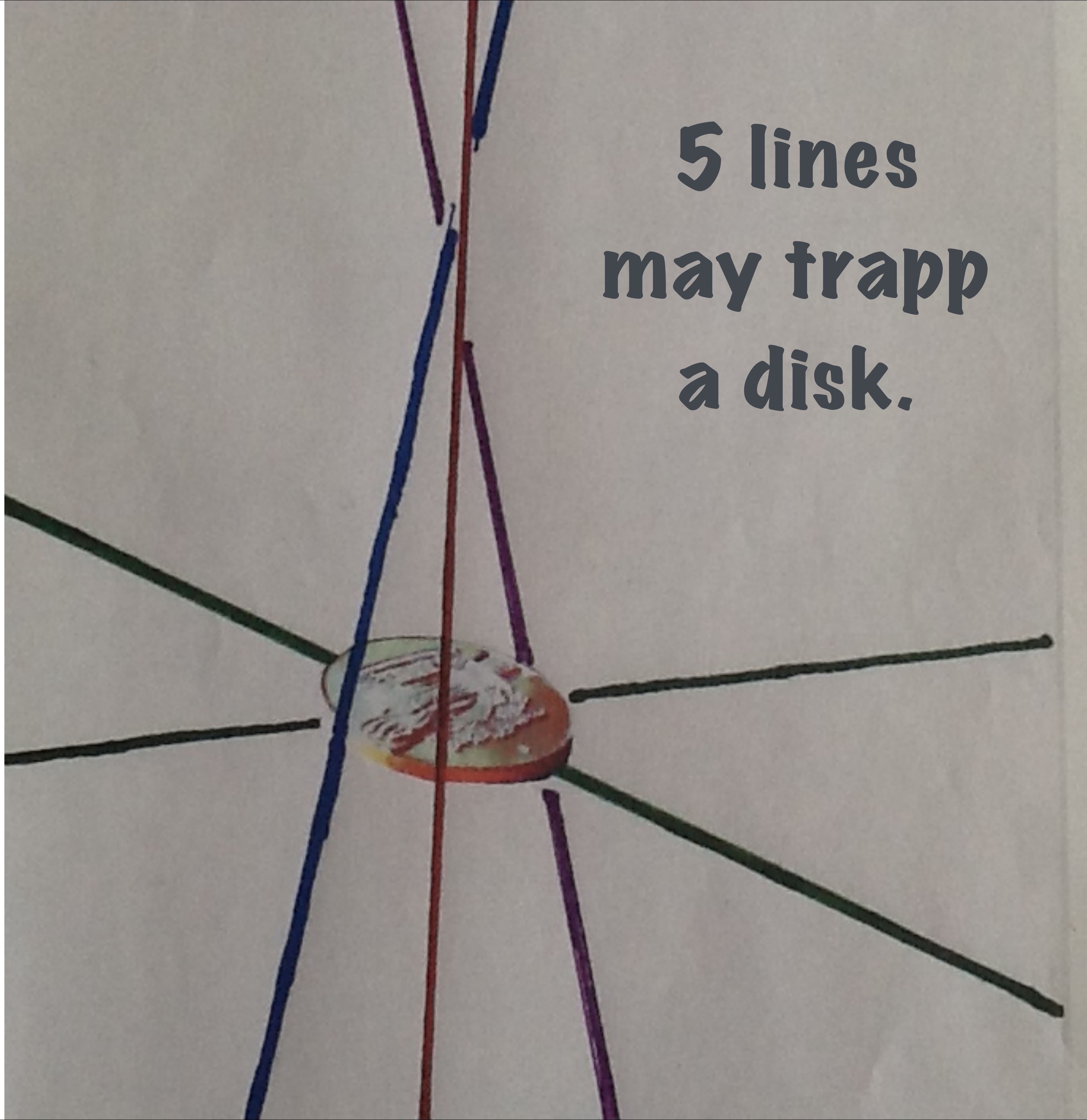

HoLdung
circle

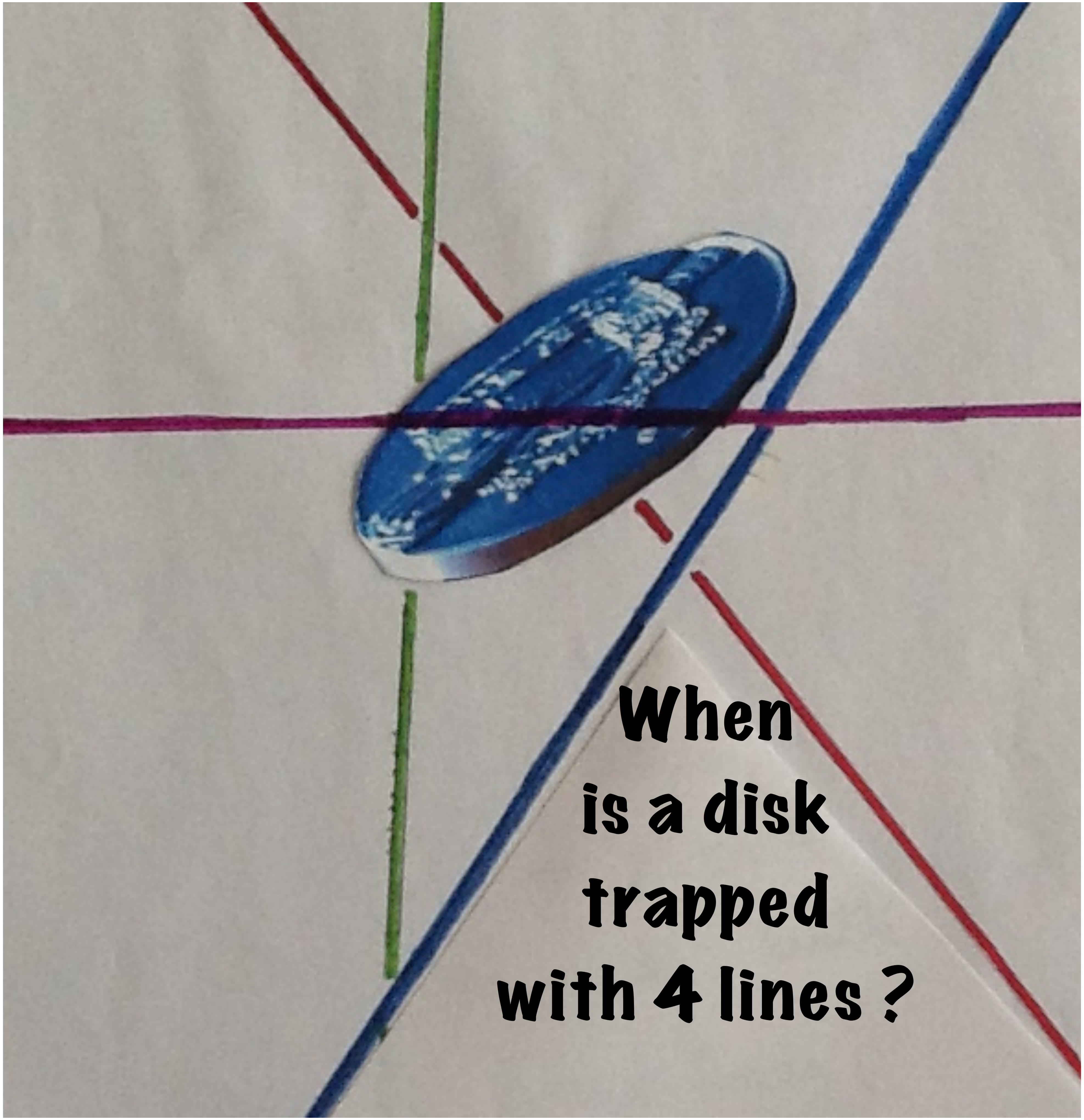
A a closed subset of euclidean 3-space.

D a planar disk.

D is trapped by A if
D cannot continuously moved through infinity
without its relative interior intersecting transversally A


3 lines never
trapp a disk.


5 lines
may trapp
a disk.


When
is a disk
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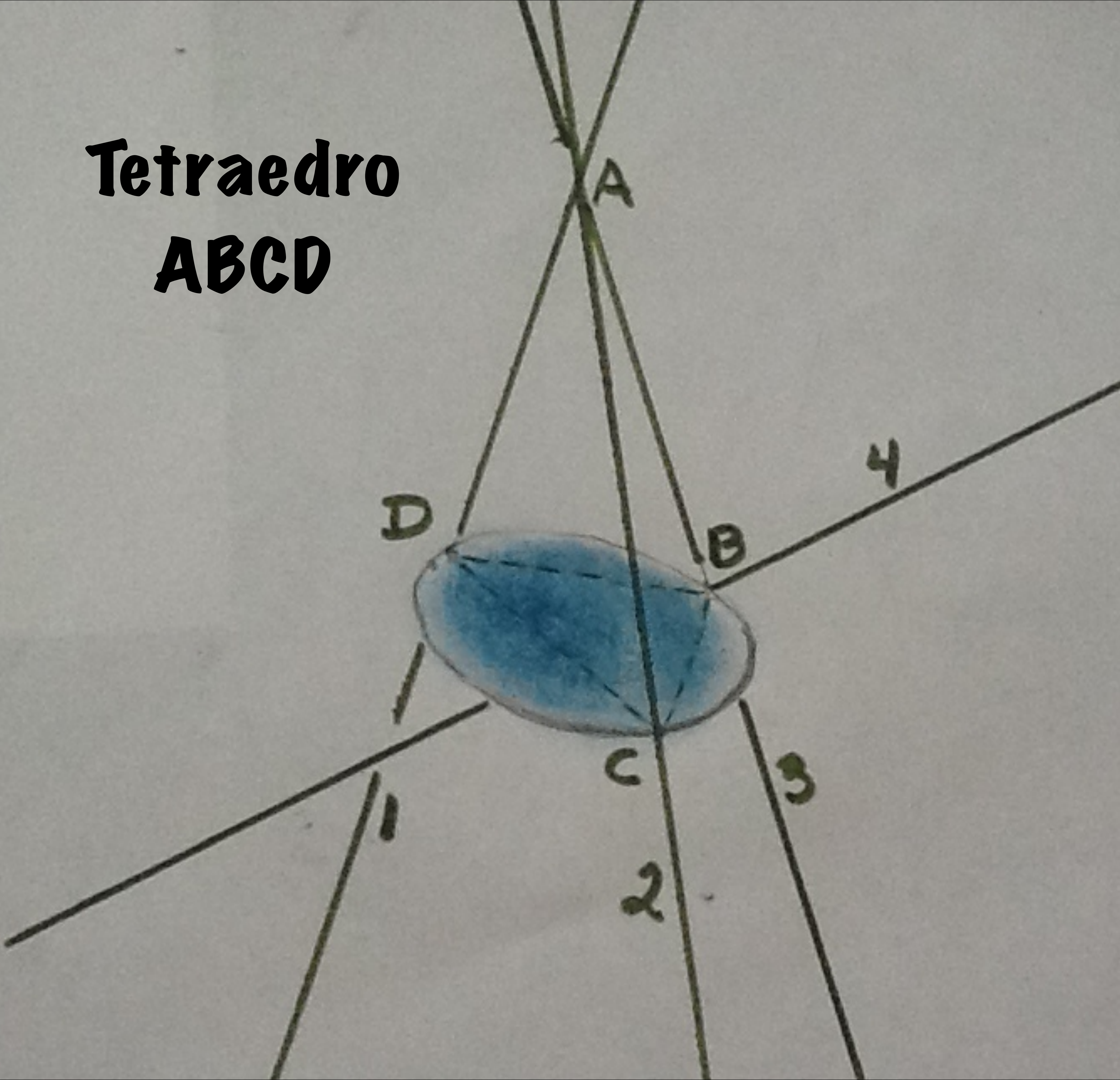
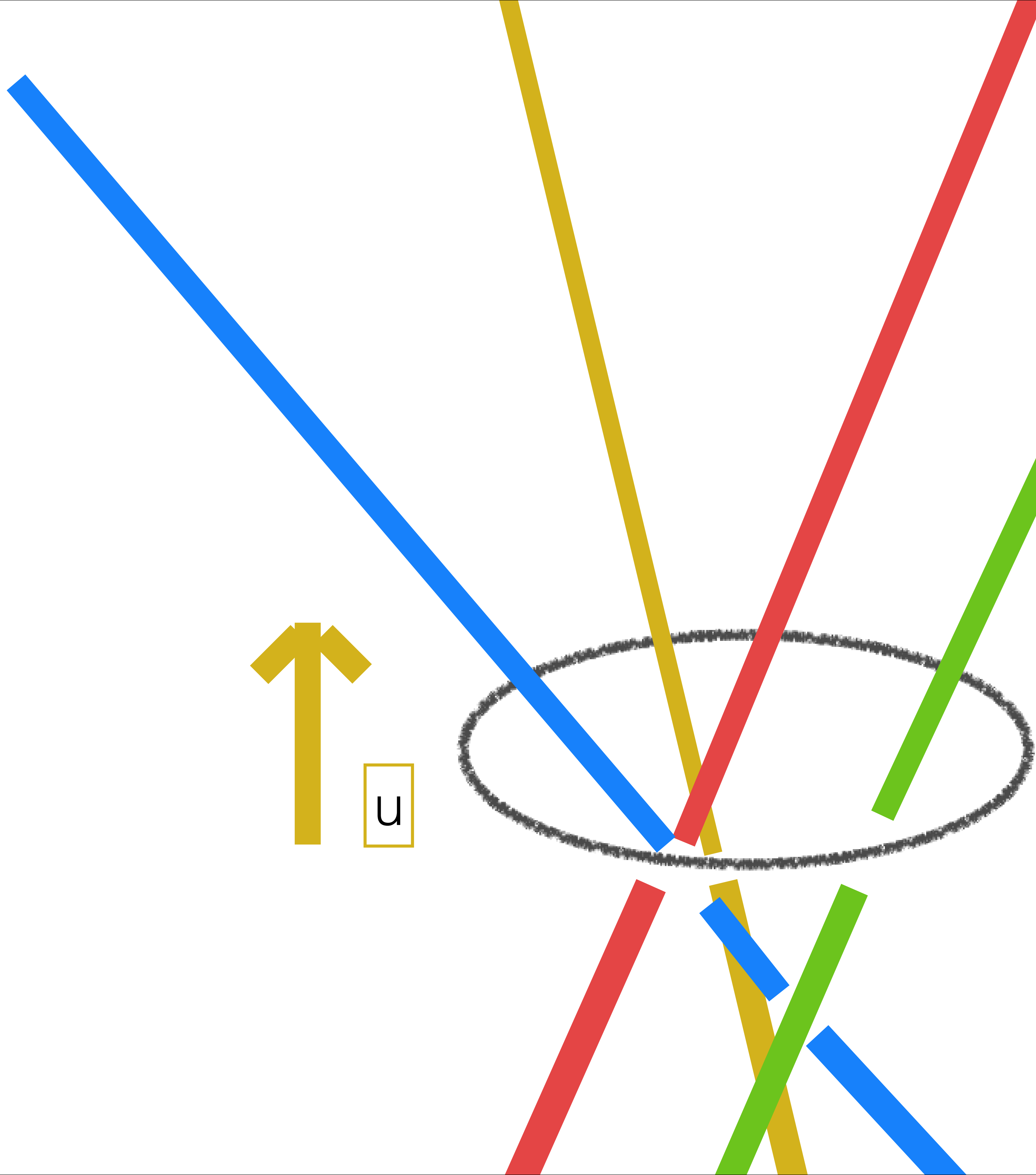
with 4 lines ?

Tetraedro ABCD





Find the smallest disk

Orthogonal to the direction u

Intersecting all lines

$\mathcal{L}$ is a finite collection of lines in $\mathbb{R}^3$

$$\varphi_{\mathcal{L}} : S^2 \longrightarrow \mathbb{R}$$

$\varphi_{\mathcal{L}}(u) =$ diameter of the smallest disk, orthogonal to u, intersecting all lines of $\mathcal{L}$

for every $u \in S^2$

Essentially

A disk is trapped by

$\mathcal{L} = \{L_1, L_2, L_3\ L_4\}$

four lines

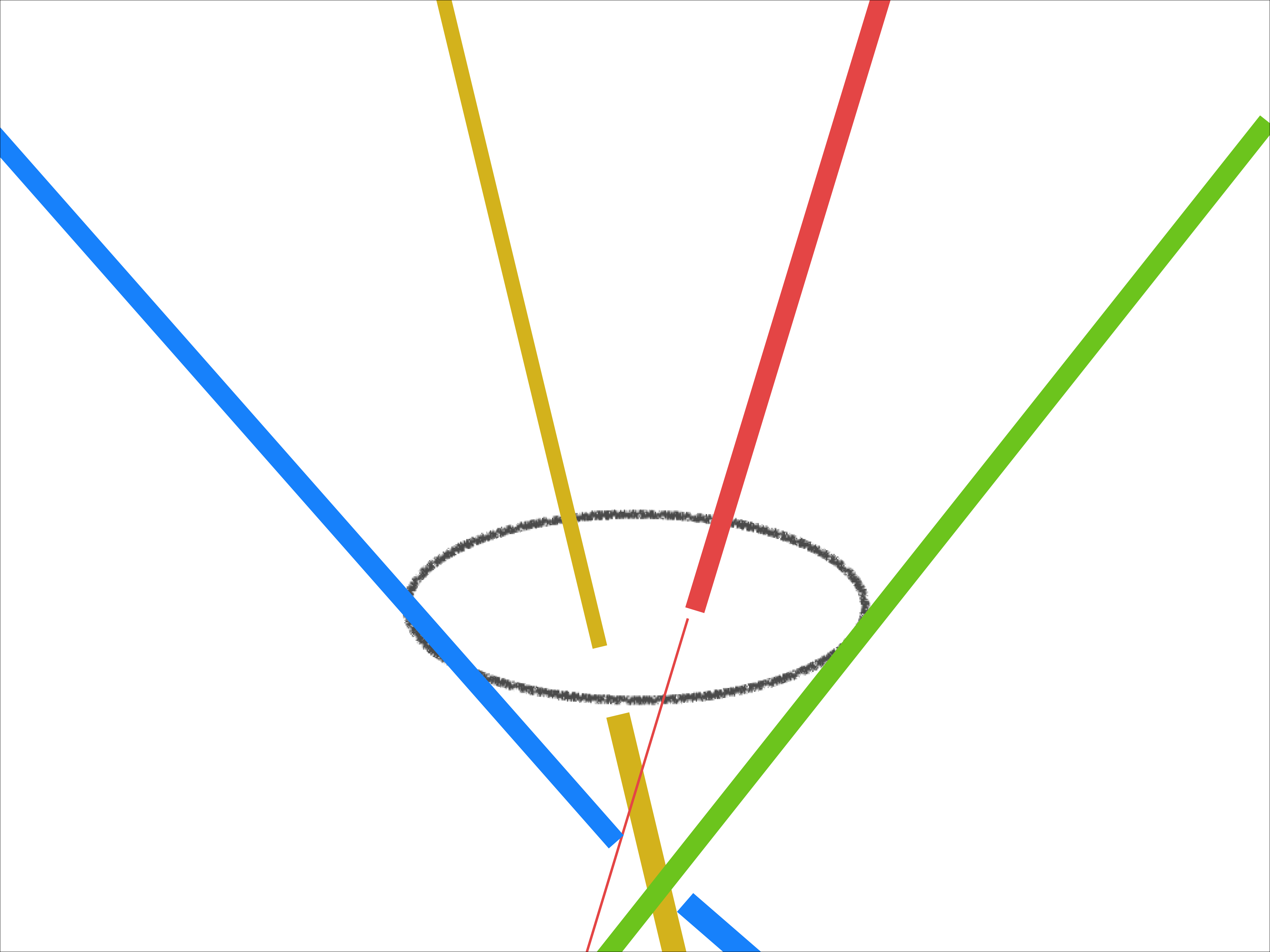
if and only if

$\varphi_{\mathcal{L}}$ has a <u>local</u> <u>maximum</u>

We need that this local maximum condition arrives from our 4 lines and it does not arrive from the local maximatily function of three of my lines.

$$\varphi_{\mathcal{L}} = \psi$$

$$\varphi_{\mathcal{L}-\mathcal{L}_i} = \psi_i$$



Theorem. *Suppose there is a disk of diameter h trapped by the lines $\{L_1, L_2, L_3, L_4\}$.*

Then, there is a local maximum h_0 of the map Ψ, at vector $v_0 \in S^2$,

$$\text{such that for } j = 1, 2, 3, 4, \;\; \Psi_j(v_0) < h_0.$$

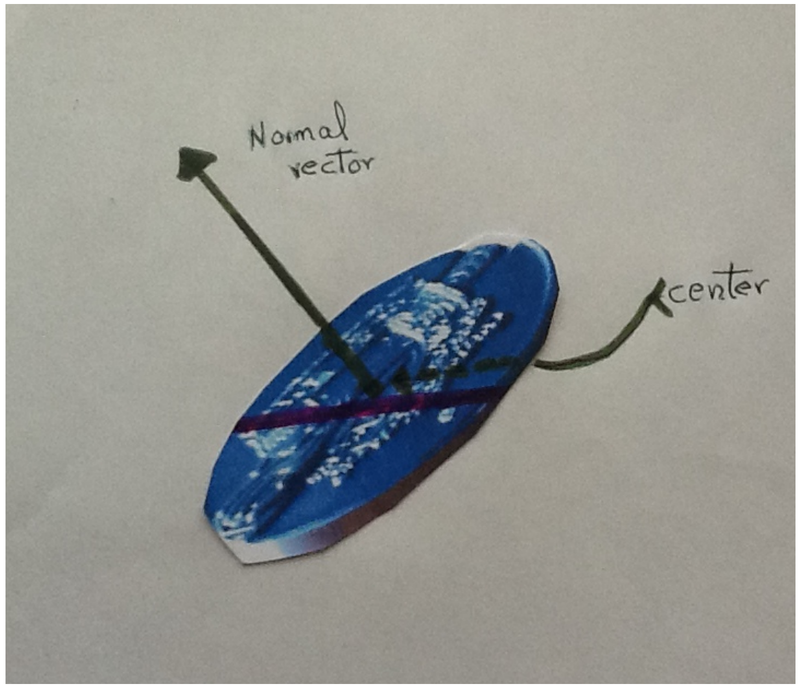
Furthermore, for every $h_1 \in [h, h_0)$, there is a disk of diameter h_1 trapped by the lines $\{L_1, L_2, L_3, L_4\}$.

Moreover, Suppose there is a local maximum h_0 of the map Ψ, at the the vector $v_0 \in S^2$,

$$\text{such that for } j = 1, 2, 3, 4, \;\; \Psi_j(v_0) < h_0.$$

Then there exists $\epsilon > 0$ such that for every $h_1 \in (h_0 - \epsilon, h_0)$, there is a disk of diameter h_1 trapped by the lines $\{L_1, L_2, L_3, L_4\}$.

S = Space of oriented unit disks



(centro, normal vector)

$$\begin{array}{c}
\mathbb{R}^3 \times S^2 \\
\shortparallel \\
\mathbb{R}^2 \times (\mathbb{R} \times S^2) \\
\shortparallel \\
\mathbb{R}^2 \times (\mathbb{R}^3 - \{0\}) \\
\cap \\
\mathbb{R}^5
\end{array}$$

L_1, L_2, L_3, L_4

$\mathcal{D}$ = Space of oriented unit disks whose interior intersect one of our four lines $\mathcal{D} \subset S \subset \mathbb{R}^5$

An oriented unit disk is trapped by

L_1, L_2, L_3, L_4

If and only if

$\mathbb{R}^5 - \mathcal{D}$

Has a bounded component

If and only if

$H_4(\mathcal{D}, \mathbb{Z}_2) \neq 0$

A disk of diameter one is trapped by the lines

L_1, L_2, L_3 and L_4

iff

$\mathbb{R}^5 - (D_1 \cup D_2 \cup D_3 \cup D_4)$ has a bounded component

iff

$H_4(D_1 \cup D_2 \cup D_3 \cup D_4 ; \mathbb{Z}_2) \neq 0$

where D_i = space of disk of diameter one that intersect the line L_i

Lemma $H_j(D_i) = 0$ for $j \geq 3$

D_i = space of disk of diameter one that intersect the line L_i

Idea: we have some control over the homology of D_i; $D_i \cup D_j$ $D_i \cap D_j$ etc

try to compute $H_4(D_1 \cup D_2 \cup D_3 \cup D_4)$ using Mayer-Vietorcas exact sequence

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & H_3([D_1 \cup D_2] \cap [D_3 \cup D_4]) \arrow[r, "\partial_2"] \arrow[d, "\partial_1"] & H_2([D_1 \cap D_2] \cap [D_3 \cup D_4]) \arrow[r, "\rho_*"] \arrow[d, "\partial_4"] & H_2(D_1 \cap (D_3 \cup D_4)) \oplus H_2(D_2 \cap (D_3 \cup D_4)) \\
0 \arrow[r] & H_2([D_1 \cup D_2] \cap [D_3 \cap D_4]) \arrow[r, "\partial_3"] \arrow[d, "k_*"] & H_1(D_1 \cap D_2 \cap D_3 \cap D_4) \arrow[r, "i_*"] \arrow[d, "j_*"] & H_1(D_1 \cap D_3 \cap D_4) \oplus H_1(D_2 \cap D_3 \cap D_4) \arrow[d] \\
0 \arrow[r] & H_2((D_1 \cup D_2) \cap D_3) \oplus H_2((D_1 \cup D_2) \cap D_4) \arrow[r, "\partial_5 \oplus \partial_6"] & H_1(D_1 \cap D_2 \cap D_3) \oplus H_1(D_1 \cap D_2 \cap D_4) \arrow[r] & {}
\end{tikzcd}
$$

Mayer Vietoris exact sequences

A disk of diameter one is trapped by the lines

L_1, L_2, L_3 and L_4

if and only if

there is α

$$\begin{tikzcd}
 & H_1(D_1 \cap D_2 \cap D_3) \\
 & H_1(D_1 \cap D_2 \cap D_4) \\
\begin{array}{c}\alpha \neq 0 \\ \in \\ H_1(D_1 \cap D_2 \cap D_3 \cap D_4)\end{array} \arrow[uur, "J_4"] \arrow[ur, "J_3"] \arrow[r, "J_2"] \arrow[dr, "J_1"] & H_1(D_1 \cap D_3 \cap D_4) \\
 & H_1(D_2 \cap D_3 \cap D_4)
\end{tikzcd}$$

$J_4(\alpha) = 0$

$J_3(\alpha) = 0$

$J_2(\alpha) = 0$

$J_1(\alpha) = 0$

In S^2

the condition

there is $\alpha \neq 0$

$H_1(D_1 \cap D_2 \cap D_3 \cap D_4)$

$$H_1(D_1 \cap D_2 \cap D_3 \cap D_4) \xrightarrow{j_4} H_1(D_1 \cap D_2 \cap D_3) \qquad j_4(\alpha) = 0$$

$$H_1(D_1 \cap D_2 \cap D_3 \cap D_4) \xrightarrow{j_3} H_1(D_1 \cap D_2 \cap D_4) \qquad j_3(\alpha) = 0$$

$$H_1(D_1 \cap D_2 \cap D_3 \cap D_4) \xrightarrow{j_2} H_1(D_1 \cap D_3 \cap D_4) \qquad j_2(\alpha) = 0$$

$$H_1(D_1 \cap D_2 \cap D_3 \cap D_4) \xrightarrow{j_1} H_1(D_2 \cap D_3 \cap D_4) \qquad j_1(\alpha) = 0$$

Means

Ψ has a local maximum

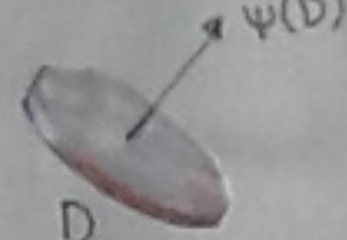
$\psi(v_0) = 1$

$\psi_i(v_0) < 1 \qquad i = 1, 2, 3, 4.$

Now we have to see the algebraic condition on the sphere S^2

Space of oriented disks $\xrightarrow{\Psi} S^2$

$D \longmapsto \Psi(D) = $ normal vector $\in S^2$



Lemma $\Psi : \bigcap_{i=1}^{k} D_i \longrightarrow \Psi\left(\bigcap_{i=1}^{k} D_i\right)$

is a homotopy equivalence

Proof the fibers $\Psi^{-1}(x)$ are contractible.

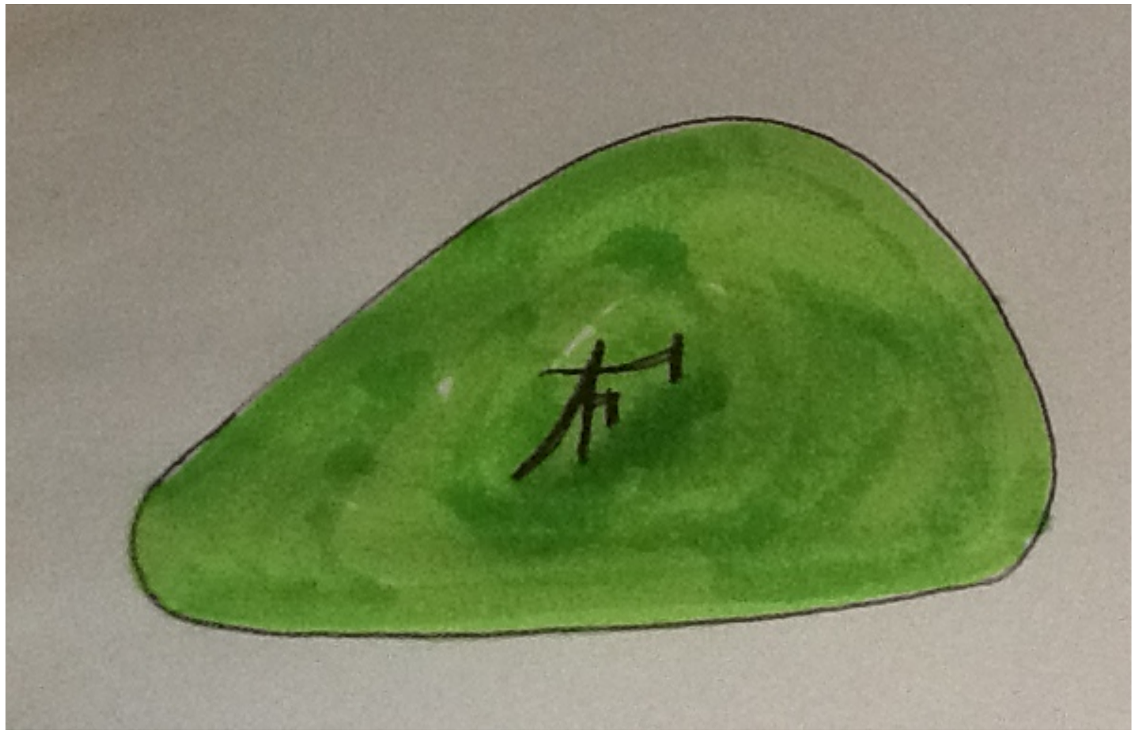
The projection of alpha to the 2-sphere
Give rise to a curve that surrounds a
local maximum of our function

This is concluded using elementary
ideas of Morse Theory on the
2-sphere.

$$\psi^{-1}(0\ h) = \Psi(D_1 \cap D_2 \cap D_3 \cap D_4)$$

singular point

Let F be a convex figure in the plane



When is a disk with the shape of F trapped by four lines ?

In particular, When is a triangle or a planar cuadrilateral figuré trapped with four lines ?

The space of disk are now all congruent copies of $\mathcal{F}$, that is:

$$S = R^3 \times SO(3)$$

L_1, L_2, L_3, L_4

D = Space of disks with the shape of F whose interior intersect one of our four lines

D is contained in S

A disk with the shape of is trapped by

L_1, L_2, L_3, L_4

If and only if

$H_4(D, Z_2) \neq 0$

$\mathcal{L}$ is a finite collection of lines

F is a convex plane figure in the plane contained the origin

$$\varphi_{\mathcal{L}} : SO(3) \longrightarrow \mathbb{R}$$

$\varphi_{\mathcal{L}}(\phi) =$ diameter of the smallest translated copy of $\phi(F)$ intersecting the lines of $\mathcal{L}$

for every $\phi \in SO(3)$

Essentially

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Furthermore, for every $h_1 \in [h, h_0)$, *there is a disk of diameter* h_1 *trapped by the lines* $\{L_1, L_2, L_3, L_4\}$.

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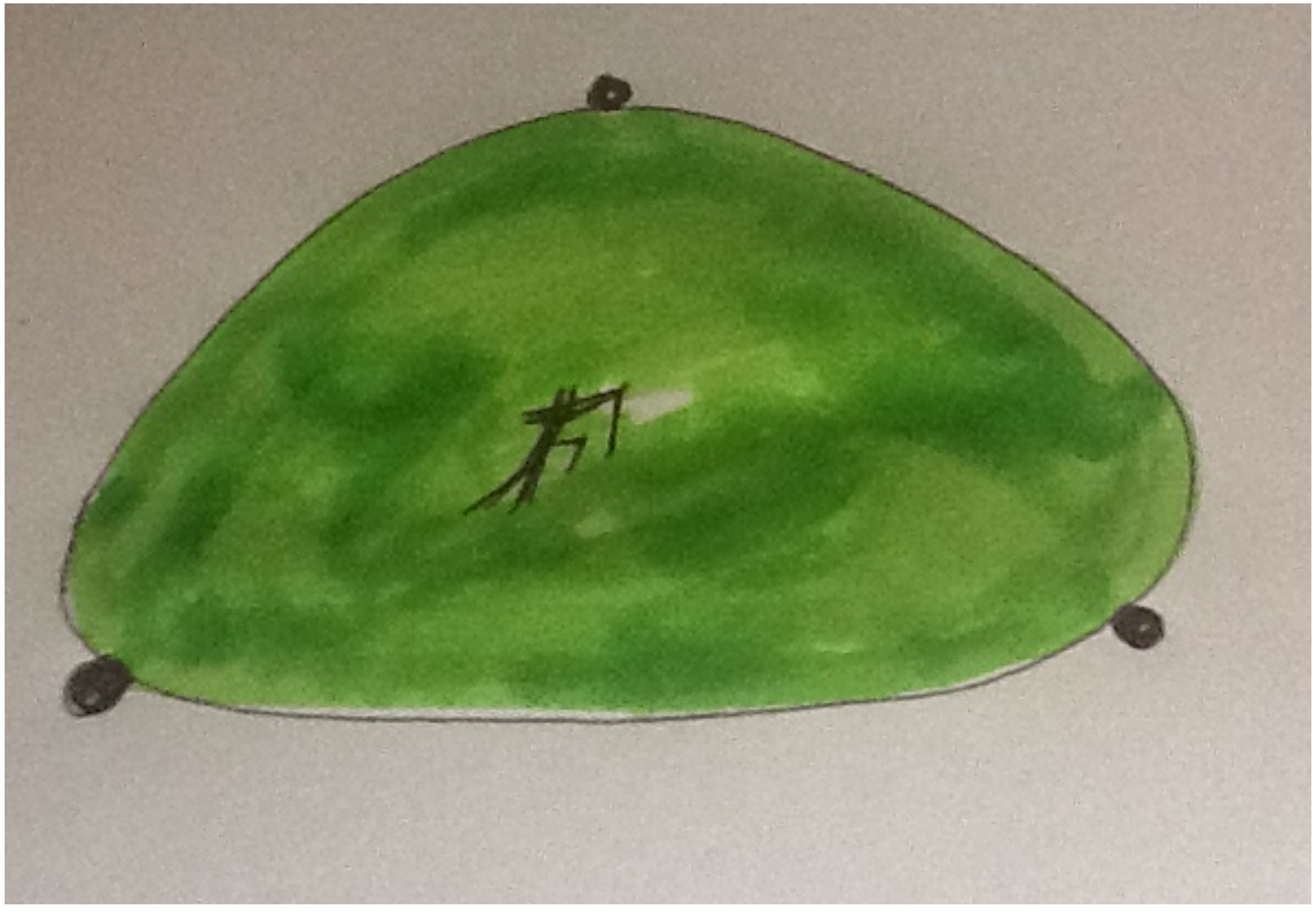
Then there exists $\epsilon > 0$ *such that for every* $h_1 \in (h_0 - \epsilon, h_0)$, *there is a disk of diameter* h_1 *trapped by the lines* $\{L_1, L_2, L_3, L_4\}$.

If a disk trapped by four lines Then there is a triangle or a cuadrilateral trapped by four lines.

If a disk trapped by four lines Then there is a triangle or a cuadrilateral immobilized by four lines.

Immobilization of Convex Figures in the plane.

A collection of points $\mathcal{H}$ on the boundary of a plane figure $\mathcal{F}$ is said tomimmobilize $\mathcal{F}$ if any small rigid movement of $\mathcal{F}$ causes on point in $\mathcal{H}$ to penetrate the interior of $\mathcal{F}$



$a_1 r_1 k_1 + a_2 r_2 k_2 + a_3 r_3 k_3 < 1$

immobilize

$a_1 r_1 k_1 + a_2 r_2 k_2 + a_3 k_3 > 1$

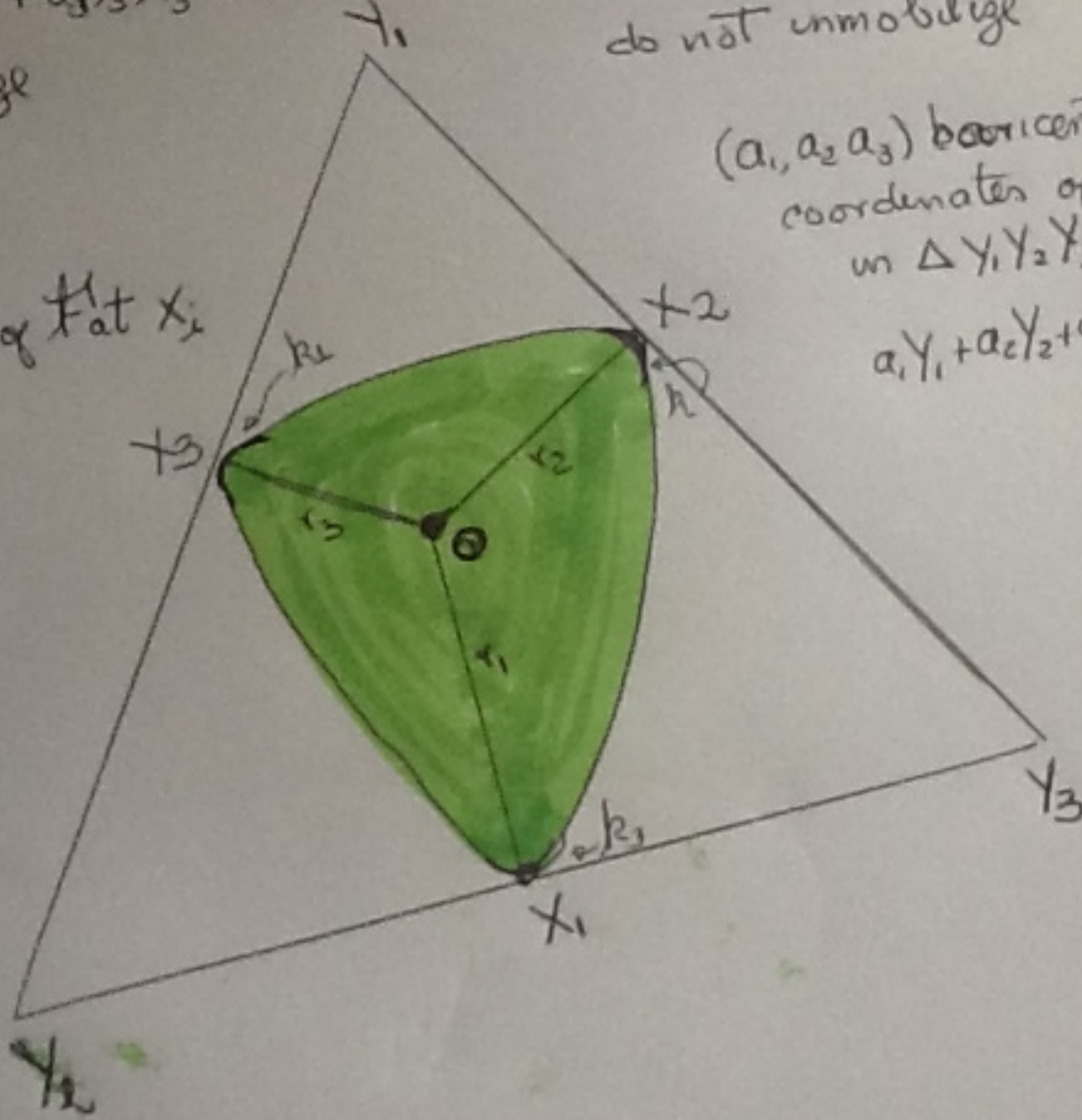
do not immobilize

$(a_1, a_2 a_3)$ baricentric coordenates of O in $\Delta Y_1 Y_2 Y_3$

$r_i = |X_i|$

k_i curvature of K at X_i

$a_1 Y_1 + a_2 Y_2 + a_3 Y_3 = 0$



There is a formula that tell us when four points in the boundary of a three dimensional convex body immobilize it. This formula is expressed in terms of cuadratic forms.

Work in Progress

To find a formula in terms of cuadratic forms that tell us when a cuadrilateral plane figure is immobilized by 4 lines.