Stability theorems

Tamás Szőnyi

ELTE, CAI HAS

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A prototype of a stability theorem:

Turán graph ($T(n,p)$), extremal graph with n vertices, not containing K_{p+1}. Number of edges of $T(n,p) := t(n,p)$.

Stability of the Turán graph: $\forall \varepsilon > 0$, $\exists \delta$ and $n(\varepsilon)$, so that:
if the graph G_n on n vertices, $n > n(\varepsilon)$, does not contain K_{p+1}, and the number of edges $> t(n,p) - \delta n^2$
$\Rightarrow$ G_n can be obtained from $T(n,p)$, by changing (adding or deleting) $\leq \varepsilon n^2$ edges. (ERDŐS, SIMONOVITS)

A special case

Theorem (Mantel)

A K_3-free (simple) graph on n vertices has at most $m \leq n^2/4$ edges.

Let $\{x, y\}$ be an edge. Then $N(x) \cap N(y) = \emptyset$, hence $\deg(x) + \deg(y) \leq n$. Summing for all edges we get

$$\sum_x (\deg(x))^2 \leq nm.$$

As $\sum_x (\deg(x))^2 \geq n(\sum_x (\deg(x)/n)^2 = 4m^2/n$ we get Mantel's result.

Towards a stability theorem: 1) In case of equality the graph is bipartite with classes of size $n/2, n/2$ (or $(n+1)/2, (n-1)/2$.

A special case: stability version

2) If $\deg(x) + \deg(y) = n$, then the graph is bipartite. Hence if $m > n(n-1)/4$, then the graph is bipartite. Indeed, $N(x)$ and $N(y)$ are the two classes (these are always independent sets).

3) What happens if $\deg(x) + \deg(y) = n - 1$?
If the remaining point is connected to a points in $N(x)$ and b points in $N(y)$ then we loose ab edges and win $a + b$ edges. Maximum no. of edges: $a = 1$ (or $b = 1$): $1 + (n-1)^2/4$ if n is odd, $1 + ((n-1)^2 - 1)/4$, if n is even.

4) Otherwise, $m \leq n(n-2)/2$.

A special case: Erdős–Simonovits

Assume we find two adjacent points with $\deg(x) + \deg(y) \geq (1-\gamma)n$. Then $N(x)$ and $N(y)$ induce a large bipartite subgraph. To get this from the original graph, we have to modify at most $(\gamma - \gamma^{/}2)n^2$ edges.

If there is no such pair (x, y) then there are at most $(1-\gamma)n^2/4$ edges. Hence for $\delta = \gamma/4$, $\varepsilon = (4\delta - 8\delta^2)$ is good if n is large enough.

More general stability theorems take into account the number of copies of K_{p+1}'s in G: LOVÁSZ, SIMONOVITS

A classical result in extremal combinatorics

ERDŐS-KO-RADO, 61: If $\mathcal{F}$ is a k-uniform intersecting family of subsets of an n element set S, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ provided $2k \leq n$.

If $2k+1 \leq n$, then equality holds if and only if $\mathcal{F}$ is the family of all subsets containing a fixed element $s \in S$. In other words, $\tau(\mathcal{F}) = 1$. What if $\tau \geq 2$?

This was answered by HILTON, MILNER, see next slide.

Non-uniform version (exercise!): $|\mathcal{F}| \leq 2^{n-1}$.

Proof: (matching: $X, S \setminus X$) At most one can be in $\mathcal{F}$.

A corresponding stability result

Theorem (Hilton-Milner, 67)

Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family with $k \geq 3$, $2k+1 \leq n$ and $\tau(\mathcal{F}) \geq 2$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. The families achieving that size are

- *for any k-subset F and $x \in [n] \setminus F$ the family $\mathcal{F}_{HM} = \{F\} \cup \{G \in \binom{[n]}{k} : x \in G, F \cap G \neq \emptyset\}$,*
- *if $k = 3$, then for any 3-subset S the family $\mathcal{F}_3 = \{F \in \binom{[n]}{3} : |F \cap S| \geq 2\}$.*

Vector space analogues of EKR theorem: HSIEH, GREENE-KLEITMAN, FRANKL-WILSON

Vector space analogues of HM: BLOKHUIS, BROUWER, CHOWDHURY, FRANKL, MUSSCHE, PATKÓS, SZT for $n \geq 2k+1$

BLOKHUIS, BROUWER, SZT: for $n = 2k$

Another extremal problem in graph theory

A bipartite graph $G = (A, B, E)$ is $K_{\alpha,\beta}$*-free* if it does not contain α nodes in A and β nodes in B that span a subgraph isomorphic to $K_{\alpha,\beta}$. We call $(|A|, |B|)$ the *size* of G. The maximum number of edges a $K_{\alpha,\beta}$-free bipartite graph of size (m, n) may have is denoted by $Z_{\alpha,\beta}(m, n)$, and is called a *Zarankiewicz number*.

Other formulation: Max. no. of 1's in an $m \times n$ 0/1 matrix without having an $\alpha \times \beta$ submatrix with only 1's.

Results for general α, β: KŐVÁRI–T. SÓS– TURÁN, KOLLÁR–RÓNYAI–SZABÓ, ALON–RÓNYAI–SZABÓ

Constructions and bounds: GUY, BROWN, FÜREDI and the authors above.

We shall focus on the case $\alpha = \beta = 2$.

Reiman's theorem

Theorem (Reiman)

Let $G = (A, B, E)$ be a $K_{2,2}$-free bipartite graph of size (n, n). Then the number of edges in G,

$$|E| \leq \frac{n}{2}(1 + \sqrt{4n-3}).$$

Equality holds if and only if $n = k^2 + k + 1$ for some k and G is the incidence graph of a projective plane of order k.

For bipartite graphs of size (m, n) the corresponding bound is

$$|E| \leq \frac{1}{2}(n + \sqrt{n^2 + 4nm(m-1)}),$$

and in case of equality we have a $2-(n, k, 1)$ Steiner system with $m = v(v-1)/k(k-1)$ blocks.
Later REIMAN and ROMAN found that extremal constructions with larger α, β are related to block designs.

Roman's bound

Theorem (Roman's bound)

Let $G = (A, B, E)$ be a $K_{s,t}$-free bipartite graph of size (m, n), and let $p \geq s - 1$. Then the number of edges in G,

$$|E| \leq \frac{(t-1)}{\binom{p}{s-1}} \binom{m}{s} + n \cdot \frac{(p+1)(s-1)}{s}.$$

Equality holds if and only if every vertex in B has degree p or $p+1$ and every s-tuple in A has exactly $t-1$ common neighbours in B.

Stability version?

What would be a stability version of Reiman's theorem?

Several possibilities (all too general):

1. If a bipartite graph has somewhat less edges than projective planes (or designs), the it can be embedded in a plane (or design).
2. If m and n are close to the parameters of a projective plane (or a design) then the extremum in Zarankiewicz' problem can be obtained by deleting/adding vertices (and perhaps edges) from a plane (or a design).

We would need incidence structures close to a projective/affine plane or a design. Candidates: linear spaces, partial projective planes etc.

The corresponding embeddability results are typically not strong enough.

Stability of projective planes

Result (Metsch)

Let $n \geq 15$, $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an incidence structure with $|\mathcal{P}| = n^2 + n + 1$, $|\mathcal{L}| \geq n^2 + 2$ such that every line in $\mathcal{L}$ is incident with $n + 1$ points of $\mathcal{P}$ and every two lines have at most one point in common. Then a projective plane Π of order n exists and $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ can be embedded into $\mathcal{P}$. □

Lemma

Let $n \geq 15$, $G = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an incidence graph with $|\mathcal{P}| = n^2 + n + 1$, $|\mathcal{L}| \geq n^2 + 2$ such that every line in $\mathcal{L}$ is incident with at least $n + 1$ points of $\mathcal{P}$, and every two lines have at most one point in common. Then a projective plane Π of order n exists, and $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ can be embedded into $\mathcal{P}$; specially, every line in $\mathcal{L}$ is incident with exactly $n + 1$ points of $\mathcal{P}$.

Consequence for the Zarankiewicz problem

Theorem (Damásdi, Héger, SzT)

Let $n \geq 15$, $c \leq n/2$. Then

$$Z_{2,2}(n^2+n+1-c, n^2+n+1) \leq (n^2+n+1-c)(n+1).$$

Equality holds iff a projective plane of order n exists. Moreover, graphs giving equality are subgraphs of the incidence graph of a projective plane of order n.

Is it true that $Z_{2,2}(n^2+1, n^2+n+1) \leq (n^2+1)(n+1)$ for every large enough n?

Subgraph-closed families

Let $\mathcal{F}(m,n) = \{G = (A,B) \in \mathcal{F} \colon |A| = m, |B| = n\}$, and let $ex_{\mathcal{F}}(m,n) = \max\{e(G) \colon G \in \mathcal{F}(m,n)\}$, and let $Ex_{\mathcal{F}}(m,n) = \{G \in \mathcal{F}(m,n) \colon e(G) = ex_{\mathcal{F}}(m,n)\}$. Elements of $Ex_{\mathcal{F}}(m,n)$ are called extremal.

Theorem

Let $\mathcal{F}$ be a subgraph-closed family of bipartite graphs, and let $ex_{\mathcal{F}}(m,n) \le e = a(d+1) + bd$, $0 \le a, b \in \mathbb{N}$, $m = a+b$, $d \in \mathbb{N}$. Let $c \in \mathbb{N}$.

1. *If $b = 0$, then $ex_{\mathcal{F}}(m+c,n) \le (m+c)(d+1)$.*
2. *If $b > 0$ or $ex_{\mathcal{F}}(m+1,n) \le e+d$, then $ex_{\mathcal{F}}(m+c,n) \le e+cd$.*

Moreover, in both cases equality for some $c \ge 1$ implies that equality holds for all $c' \in \mathbb{N}$, $0 \le c' < c$ as well, and any $G \in Ex_{\mathcal{F}}(m+c,n)$ induces a graph from $Ex_{\mathcal{F}}(m+c-1,n)$.

A useful embedding result for the complement of two lines in a projective plane.

Result (Totten)

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ *be a finite linear space (that is, an incidence structure where any two distinct points are contained in a unique line) with* $|\mathcal{P}| = n^2 - n$, $|\mathcal{L}| = n^2 + n - 1$, $2 \leq n \neq 4$, *and every point having degree* $n + 1$. *Then* $\mathcal{S}$ *can be embedded into a projective plane of order* n.

A consequence for the Zarankiewicz problem

Proposition (Damásdi, Héger, SzT)

Let $c \in \mathbb{N}$. Then

$$
\begin{aligned}
Z_{2,2}(n^2+c, n^2+n) &\leq n^2(n+1)+cn,\\
Z_{2,2}(n^2-n+c, n^2+n-1) &\leq (n^2-n)(n+1)+cn,\\
Z_{2,2}(n^2-2n+1+c, n^2+n-2) &\leq (n^2-2n+1)(n+1)+cn,\\
&\quad \textit{if } n \geq 4.
\end{aligned}
$$

Equality can be reached in all three inequalities if a projective plane of order n exists and $c \leq n+1$, or $c \leq 2n$, or $c \leq 3(n-1)$, respectively.
Moreover, if $c \leq n+1$, or $c \leq 2n$ and $2 \leq n \neq 4$, then graphs reaching the bound in the first two cases, can be embedded into a projective plane of order n.

What would be the real question?

What is $Z_2, 2(m, n)$ if both m and n are close to $k^2 + k + 1$?
Warning: $n = m = 8$: two examples, one from an affine plane of order 3, another a Fano plane plus one point! (They are different even from the combinatorial point of view.)

Other good question: if we have somewhat less than $\frac{n}{2}(1 + \sqrt{4n-3})$ edges in a bipartite graph of size (n, n) can one obtain it from the incidence graph of a projective plane by deleting some edges?

The previous embedding theorems (Metsch, Totten) are special cases, we don't know such a general result.

Arcs in finite planes

ARC: set of points no three of which are collinear

complete arc: maximal w.r.t. inclusion

BOSE: In a projective plane of order q an arc has AT MOST $q+2$ points. If q is odd then it has AT MOST $q+1$ points.

OVALS, HYPEROVALS

EXAMPLES: CONICS in $PG(2,q)$ (for q even an extra point can be added.)

Stability results in finite geometry: arcs

More common name: SEGRE type results

Theorem (SEGRE)

If A is arc in $\mathrm{PG}(2,q)$ with $|A| \geq q-\sqrt{q}+1$ when q is even and $|A| \geq q-\sqrt{q}/4+7/4$ when q is odd, then A is contained in an arc of maximum size (that is, in an oval or hyperoval).

Several improvements: THAS, VOLOCH, HIRSCHFELD-KORCHMÁROS

Beautiful improvement:

Theorem (HIRSCHFELD-KORCHMÁROS)

Let A be arc in $\mathrm{PG}(2,q)$ with $q-2\sqrt{q}+5<|A|<q-\sqrt{q}+1$. Then A is contained in a larger arc of size $q+2$ or $q-\sqrt{q}+1$.

Smallest blocking sets: lines
2nd smallest: Baer subplanes, BRUEN (in Π_q)
Stability of minimal blocking sets: in $\mathrm{PG}(2,q)$ small minimal blocking sets have size in certain subintervals of $[q+1, \frac{3}{2}(q+1))$, SZT, SZIKLAI
In some cases small blocking sets can be classified/characterized: BLOKHUIS, SZT, POLVERINO, POLVERINO-STORME
BLOKHUIS: In $\mathrm{PG}(2,p)$ there are no small blocking sets.
Another stability version:

Theorem (Erdős and Lovász)

A point set of size q in $\mathrm{PG}(2,q)$, with less than $\sqrt{q+1}(q+1-\sqrt{q+1})$ 0-secants always contains at least $q+1-\sqrt{q+1}$ points from a line.

Stability of small blocking sets I

For planes $\mathrm{PG}(2, q)$, with q prime.

Theorem (Weiner-SzT)

Let B be a set of points of $\mathrm{PG}(2, q)$, $q = p$ prime, that has at most $\frac{3}{2}(q+1) - \beta$ ($\beta > 0$) points. Suppose that the number of 0-secants, δ is less than $(\frac{2}{3}(\beta+1))^2/2$. Then there is a line that contains at least $q - \frac{2\delta}{q+1}$ points.

Note that for $|B| = cq$, $c \geq 1$ the bound on δ in the above theorem is $c'q^2$.

Stability of small blocking sets II

Theorem (Weiner-SzT)

Let B be a point set in $\mathrm{PG}(2,q)$, $q \geq 16$, of size less than $\frac{3}{2}(q+1)$. Denote the number of 0-secants of B by δ, and assume that

$$\delta < \min\left((q-1)\frac{2q+1-|B|}{2(|B|-q)}, \frac{1}{2}(q-\sqrt{q})^{3/2}\right). \qquad (1)$$

Then B can be obtained from a blocking set by deleting at most $\frac{\delta}{2q+1-|B|}+\frac{1}{2}$ points of it.

When $|B|$ is relatively far from q and gets closer to $\frac{3}{2}(q+1)$ then the bound on δ gets worse.

Sets of even type

Π_q: projective plane of order q

set of even type: intersects each line in an even number of points
Only exist for q even. Hence from now on

q IS EVEN

We shall concentrate on $PG(2, q)$.

More general structures (e.g. Steiner systems): TALLINI
Smallest example has $q + 2$ points: hyperoval.

Small sets of even type are close to arcs, hence we cannot expect a stability result for arcs from above.

A class of examples

KORCHMÁROS, MAZZOCCA:

$(q+t,t)$-arc of type $(0,2,t)$: set of $q+t$ points such that every line meets it in either 0,2 or t points

In $PG(2,q)$ KORCHMÁROS, MAZZOCCA: t divides q
Conjecture that for every $4|t|q$ there is such an arc

GÁCS, WEINER: t-secants are concurrent

Constructions: KM, GW infinite classes, e.g. for $t=\sqrt{q}$.

VANDENDRIESSCHE: infinite class for $t=q/4$

Sporadic ones: $q=32$, $t=4$, KEY, MCDONOUGH, MAVRON
even more examples: LIMBUPASIRIPORN

Sets of almost even type

set of almost even type: has only few odd secants
Typical example: modify ε pts of a set of even type
No. of odd secants is roughly $\delta = \varepsilon(q+1)$.

AIM: show that a set of almost even type is "typical" if δ=no. of odd-secants is small

This is a stability theorem

Further motivation: small Kakeya-sets in $\mathrm{AG}(2, q)$, q even
BLOKHUIS, DE BOECK, MAZZOCCA, STORME

Rough picture: dual of a (small Kakeya set + line of infinity) is a $(q+2)$-set having a small number of odd secants (almost all intersecting lines are 2-secants) and a special point through which there pass only 2-secants. More details later.

Our stability theorem

Theorem (Weiner-Szőnyi)

Assume that the point set $\mathcal{H}$ *in* $\mathrm{PG}(2,q)$, $16 < q$ *even, has* δ *odd-secants, where* $\delta < (\lfloor\sqrt{q}\rfloor + 1)(q + 1 - \lfloor\sqrt{q}\rfloor)$. *Then there exists a unique set* $\mathcal{H}'$ *of even type, such that* $|(\mathcal{H} \cup \mathcal{H}') \setminus (\mathcal{H} \cap \mathcal{H}')| = \lceil \frac{\delta}{q+1} \rceil$.

So "small" on the previous slides is roughly $q\sqrt{q}$ and the number of modified points is what we expect.

Application to arcs

An arc with k points has $k(q+2-k)$ odd secants (tangents).

Hence for a complete k-arc in $\mathrm{PG}(2,q)$, q even we (almost) get SEGRE's bound, namely we get

$$k \leq q - \lfloor \sqrt{q} \rfloor + 1.$$

This is sharp for q square (proved earlier with essentially the same method by WEINER).
Also applies for sets, where "almost all" lines are 2-secants.

No. of lines intersecting a point set of size $(q+2)$

The minimum no. of lines is obtained for hyperovals. The next result gives the stability of hyperovals.

Theorem (Blokhuis, Bruen)

Let $\mathcal{H}$ be a point set in $\mathrm{PG}(2,q)$, q even, of size $q+2$. Assume that the number of lines meeting $\mathcal{H}$ in at least one point is $\binom{q+2}{2}+\nu$, where $\nu \leq \frac{q}{2}$. Then $\mathcal{H}$ is a hyperoval or there exist two points P and Q, so that $(\mathcal{H} \setminus P) \cup Q$ is a hyperoval.

One cannot go further, since by deleting 2 points from a $(q+4,4)$-arc of type $(0,2,4)$ one gets a set of size $q+2$ having fewer lines meeting it in at least one point than a hyperoval ± 2 points, see the next slide.

Sets with few lines meeting them are almost arcs.

Theorem

Let $\mathcal{H}$ be a point set in $\mathrm{PG}(2,q)$, $16 < q$ even, of size $q+2$. Assume that the number of lines meeting $\mathcal{H}$ in at least one point is $\binom{q+2}{2} + \nu$, where $\nu < \frac{1}{4}(\lfloor\sqrt{q}\rfloor + 1)(q + 1 - \lfloor\sqrt{q}\rfloor)$. Then there exists a set $\mathcal{H}^*$ of even type, such that $|(\mathcal{H} \cup \mathcal{H}^*) \setminus (\mathcal{H} \cap \mathcal{H}^*)| \leq \lceil \frac{4\nu}{q+1} \rceil$.

Note that the number of points that have to be modified is again what we expect.

Small Kakeya sets

How to obtain a small dual Kakeya set?

1. Start from a small set of even type
2. Delete some of its points to get a $(q+2)$-set (with a special point through which there are only 2-secants)

Smallest ones: hyperoval, hyperoval ± 1 points, $(q+4,4)$-arc of type $(0,2,4)$, hyperoval ± 2 points, ...

Problem: we do not know enough about sets of even type of size $q+6$ (do they exist?) or about larger ones ...

THANK YOU

THANK YOU FOR YOUR ATTENTION