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On isometries of Finsler manifolds

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- – Finsler metrics, examples
- – isometries of Finsler manifolds
- – the group of isometries
- – characterizations of isometries with area and angle
- – Finsler manifolds with many isometries
- – Weinstein theorem for Finsler manifolds

The notion of a Finsler metric

Approach I: $\forall p \in M \quad L_p : T_pM \to R^+$ norm

- $L_p(u) \geq 0 \qquad = 0 \Longleftrightarrow u = 0$
- $L_p(\lambda(u)) = \lambda L_p(u) \qquad \lambda > 0$ positively homogeneous
- $L_p(u + v) \leq L_p(u) + L_p(v)$ convexity
- $L^2 : TM \setminus \{0\} \to R^+$ is of class C^2
- $L_p(-u) = L_p(u)$ symmetrical/ reversible

indicatrix: $\mathcal{I}_p = \{u \in T_pM \mid L_p(u) = 1\}$

Approach II: variational problem

$$\int_a^b L(x(t), \dot{x}(t))dt \longrightarrow \text{Euler-Lagrange equations}$$

. $\uparrow$ positively homogenous

Riemannian case: $L(x, \dot{x}) = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}$

Finslerian case: $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$

$g(x, y)$: Riemannian metric in the Finsler vector bundle VTM

Approach III: $d : M \times M \to R^+$ is a metric

$v \in T_pM; c : [0, 1] \to M$ with $c(0) = p$, $\dot{c}(0) = v$

$$L_p(v) = \lim_{t \to 0} \frac{d(p, c(t))}{t}$$

Example 1: Funk metric

$\Omega \subset R^n$ strictly convex

$$d(p,q) = \ln \frac{|z-p|}{|z-q|}$$

$$p + \frac{y}{L(y))} \in \partial\Omega$$

$$B^n = \Omega; \quad L(y) = \frac{\sqrt{|y|^2 - (|p|^2|y|^2 - (p,y)^2)} + (p,y)}{1-|p|^2}$$

— projectively flat

— constant negative curvature $-1/4$

— non–reversible

— Randers metric

Example 2: Hilbert metric

$$\tilde{d}(p,q) = \frac{1}{2}(d(p,q) + d(q,p)) = \frac{1}{2}\left|\ln\left(\frac{|z-p|}{|z-q|} : \frac{|v-p|}{|v-q|}\right)\right|$$

Example 3: Katok's example (1973), W. Ziller (1982)

S^2; standard Riemannian metric α
Φ_t: one parameter group of rotations
leaving the north & south poles invariant
X: Killing vector field
β: Killing form

$$L_\varepsilon(x,y) = \alpha(x,y) + \varepsilon\beta(x,y)$$

Theorem: For any irrational ε a curve c is a closed geodesic of L_ε if and only if c is a closed geodesic of α and invariant with respect to Φ_t.

Properties:

– the length of the two closed geodesics: $\dfrac{2\pi}{1+\varepsilon}$; $\dfrac{2\pi}{1-\varepsilon}$
– L_ε is a Finsler metric $\Longleftrightarrow$ $|\varepsilon| < 1$

Isometries of Finsler manifolds

(M, L) : Finsler manifold

d : the induced distance function, not necessarily reversible

The length of a curve in (M, L) is given as usual:

$$\ell(c) = \int_0^1 L(\dot{c})dt.$$

The induced distance d between $x, y \in M$ can be defined by taking the infimum of the length of all curves joining x to y:

$$d(x, y) = \inf\{\, \ell(c) \,|\, c(0) = x, c(1) = y\}$$

1. an isometry: a diffeomorphism $\phi : M \to M$ of M onto itself which preserves L:

$$L(d\phi(u)) = L(u) \qquad \forall\, u \in TM$$

2. an isometry: a mapping $\phi : M \to M$ of M onto itself which preserves the distance between each pair of points:

$$d(\phi(x), \phi(x)) = d(x, y) \qquad \forall\, x, y \in M$$

[Deng, Shaoqiang and Hou, Zixin: The group of isometries of a Finsler space. Pacific J. Math. 207 (2002), no. 1, 149–155] generalizes the Myers-Steenrod theorem in Riemannian geometry:

the two definitions are equivalent.

Theorem. Let $x \in M$ and $B_x(r)$ be a tangent ball of $T_x(M)$ such that $\exp_x$ is a C^1 diffeomorphism from $B_x(r)$ onto $\mathcal{B}_x^+(r)$. For $A, B \in B_x(r)$, $A \neq B$, let $a = \exp_x A, b = \exp_x B$. Then

$$\frac{L(x, A-B)}{d(a,b)} \to 1$$

as $(A, B) \to (0, 0)$.

Theorem. Let $\|\cdot\|_1$, $\|\cdot\|_2$ be two Minkowski norms on R^n. Let ϕ be a mapping of R^n into itself such that $\|\phi(A) - \phi(B)\|_2 = \|A - B\|_1$, $\forall A, B \in R^n$. Then ϕ is a diffeomorphism.

Corollary. Let (M, L) be a Finsler space and ϕ be a distance-preserving mapping of M onto itself. Then ϕ is a diffeomorphism.

Theorem. [Deng, Hou, 2002] The group of isometries $I(M)$ is a Lie transformation group. The isotropy subgroup $I_x(M)$ is compact.

Area in Minkowski spaces

$(\mathbb{R}^n, L)$: Minkowski space
$\mathcal{B} = \{v \in \mathbb{R}^n \ : \ L(v) < 1\}$: Minkowski ball

Minkowski measure of $D \subset \mathbb{R}^n$:

$$\|D\|_M = \frac{\pi \|D\|_E}{\|\mathcal{B}\|_E}$$

independent of $\| \cdot \|_E$

Angles in Finsler geometry

Finsler angle of Finsler vectors; $U, V \in V_u TM$:

$$\sphericalangle_F(U,V) = \text{arc}\cos \frac{g_u(U,V)}{\sqrt{g_u(U,U)}\sqrt{g_u(V,V)}}$$

Minkowski angle of tangent vectors, rays in the tangent spaces
u, v: non-parallel vectors in $T_x M$;
Σ: generated linear space by u, v;
$\mathcal{B}^2 = \Sigma \cap \mathcal{B}$; $D = \text{conv}\,(u,v) \cap \mathcal{B}^2$

$$\sphericalangle_M(u,v) = \epsilon 2\|D\|_M, \qquad \epsilon = \pm 1$$

Properties: additive, symmetric; the measure of straight angle is π iff L is absolutely homogeneous (reversible).

Observation. $\phi : (M, L_1) \to (\bar{M}, L_2)$ is an isometry if and only for indicatrices

$$d\phi(\mathcal{I}_p) = \bar{\mathcal{I}}_{\phi(p)} \qquad \forall p \in M.$$

$$L_2(d\phi(u)) = L_2(L_1(u) d\phi(\frac{u}{L_1(u)})) = L_1(u) L_2(d\phi(\frac{u}{L_1(u)})) = L_1(u).$$

Theorem. [Tamássy, 2007)]
A diffeomorphism $\phi : (M, L_1) \to (\bar{M}, L_2)$ is an isometry if and only if $d\phi$ preserves the 2-dimensional area and the Minkowski angle.

Proof. Necessity: $d\phi$ is linear $\Rightarrow$ preserves the ratio of areas :

$$\|d\phi(D)\|_{\bar{M}} = \frac{\pi \|d\phi(D)\|_E}{\|\bar{\mathcal{B}}^2\|_E} = \frac{\pi \|D\|_E}{\|\mathcal{B}^2\|_E} = \|D\|_M.$$

Sufficiency. Suppose: $\phi : (M, L_1) \to (\bar{M}, L_2)$ diffeomorhism; preserves area and angle. Let $\widehat{\mathcal{B}}_p = (d\phi)^{-1}(\bar{\mathcal{B}}_{\phi(p)})$.

If $\widehat{\mathcal{I}}_p \neq \mathcal{I}_p$, then there are two nearby rays u, v such that

$$\mathrm{conv}\,(u,v) \cap \mathcal{B}_p \subset \mathrm{conv}\,(u,v) \cap \widehat{\mathcal{B}}_p,$$

however

$$\|\mathrm{conv}\,(u,v) \cap \mathcal{B}_p\|_M \overset{angle}{=} \|\mathrm{conv}\,(d\phi(u), d\phi(v)) \cap \bar{\mathcal{B}}_{\phi(p)}\|_{\bar{M}} \overset{area}{=} \|\mathrm{conv}\,(u,v) \cap \widehat{\mathcal{B}}_p$$

Remark: In this case the Finsler angle is preserved, too.

H. C. Wang, J. London Math. Soc. **22** (1947):

$n \neq 4, \quad \dim I^F(M) > \frac{1}{2}n(n-1)+1 \implies (M, L)$ is Riemannian

Ku Chao-Hao, Sci. Records N.S. **1** (1957), 215– 218.
A. I. Egorov, Gos. Ped. Inst. Ucen. Zap. (1974), 17–21.

There exist non-Riemannian Finsler spaces with

$$\dim I^F(M) = \frac{1}{2}n(n-1)+1.$$

[Szabo, Z. I. Generalized spaces with many isometries. Geom. Dedicata 11 (1981), no. 3, 369-383.]:

Study of all the non-Riemannian Finsler spaces having a group of motions of the largest order.

Theorem 1. If (M, L) is a non-Riemannian Finsler space of dimension $n > 4$ and its group of motions $I(M)$ is of order $n(n-1)/2 + 1$, it must be of one of the following types:

(1) (M, L) is a symmetric Berwald space which is the non-Riemannian Cartesian product of Riemannian spaces U [resp. V], where $U = R$, S^1 and $V = R^{n-1}, S^{n-1}, H^{n-1}, P^{n-1}(R)$,

(2) (M, L) is a BLF^n-space.

Theorem 2. Every BLF^n space ($n \geq 2$) is a non-Berwaldian Wagner space which is conformal to a Minkowski space.

H^n: hyperbolic space

$G = \{$isometries of H^n leaving S and $S^\star$ invariantly$\}$

G_p^0 isotropy group at $p \in H^n$

$r : (0, 2\pi] \to \mathbb{R}$

$(\varphi, r(\varphi))$ indicatrix of a Minkowski (non-Euclidean) norm

$g^\star$ Riemannian metric tensor of H^n

$\|X\| = \sqrt{g^\star(X, X)}$

$L(X) = r(\text{arc } \tan\frac{g^\star(N,X)}{\|X - g^\star(N,X)N\|})\|X\|$

Alan Weinstein (1968):

Let f be a an isometry of a compact oriented Riemannian manifold M. Suppose that M has positive sectional curvature and that f preserves the orientation of M if the dimension is even, and reverses if it is odd. Then f has a fixed point: $f(p) = p$.

Weinstein's Theorem for Finsler manifolds: (Kozma & Peter, 2006)

Let f be an isometry of a compact oriented positively homogeneous Finsler manifold M of dimension n. If M has positive flag curvature and f preserves the orientation of M for n even and reverses the orientation of M for n odd, then f has a fixed point.

flag curvature:

$$K(y,V)=\frac{g_y(R(V,y)y,V)}{g_y(y,y)g_y(V,V)-g_y^2(y,V)}$$

second variation formula:

Consider now the variation of σ given by

$$\Sigma:(-\epsilon,\epsilon)\times[0,\ell]\rightarrow M$$

$$\begin{aligned}\frac{d^2\ell_\Sigma}{ds^2}(0) \;&=\; \int_0^\ell\{g_{\dot\sigma}(\nabla_{\dot\sigma}U,\nabla_{\dot\sigma}U)-g_{\dot\sigma}(R_{\dot\sigma}(U),U)\}dt\\ &+g_{\dot\sigma(\ell)}(\kappa_\ell(0),\dot\sigma(\ell))-g_{\dot\sigma(0)}(\kappa_0(0),\dot\sigma(0))\\ &+\mathbf{T}_{\dot\sigma(0)}(U(0))-\mathbf{T}_{\dot\sigma(\ell)}(U(\ell))\end{aligned}$$

where $T=\dot\sigma$ and U are the tangential and transversal vector fields, resp; of the variation Σ.

Proof:

Step 1:

Suppose that the isometry f has no fixed points:
$f(x) \neq x$ for all $x \in M$.
Since the manifold M is compact, the function $h : M \to \mathbb{R}$, given by $h(x) = d(x, f(x))$ attains its minimum at a point $x \in M$: $h(x) > 0$.
The completeness of the manifold M implies that there exists a minimizing normalized geodesic $\sigma : [0, \ell]$ joining x and $f(x)$.
Show that the curves formed by σ and $f \circ \sigma$ form a geodesic.
Then $df_x(\sigma'(0))) = \sigma'(\ell))$.

Step 2:

Find a unit parallel vector field $E(t)$ which is $g_{\dot\sigma(t)}$–orthogonal complement of $\dot\sigma(t)$.
Then $df_x(E(0)) = E(\ell))$.

Step 3:

Construct a variation Σ of σ given by

$$\Sigma : (-\epsilon, \epsilon) \times [0, \ell] \to M$$
$$\Sigma(s,t) = \exp_{\sigma(t)}(sE(t)),\ s \in (-\epsilon, \epsilon),\ t \in [0, \ell].$$

Then

$$U(t) = \frac{\partial}{\partial s} \exp_{\sigma(t)}(sE(t))|_{s=0} = E(t),$$

so the transversal vector of the variation Σ is parallel transported along σ.

Step 4:

The second variation formula reduces to:

$$\frac{d^2\ell_\Sigma}{ds^2}(0) = -\int_0^\ell g_{\dot\sigma}(R(U,\dot\sigma)\dot\sigma, U)dt < 0,$$

which contradicts the minimality of the curve σ, which joins x and $f(x)$.
Therefore $d(x, f(x)) > 0$ is impossible.

Chang Wan Kim (2007, J. Math. Kyoto):
M is oriented Finsler manifold with k-th Ricci curvature $\geq k$.
f is an isometry satisfying $d(x, f(x) > \pi\sqrt{(k-1)k}$.
(1) If M is even dimensional, then f reverses the orientation.
(2) If M is odd dimensional, then f is orientation preserving.

Killing vector field $X \in \mathfrak{X}(M)$ of (M, L): if any local one-parameter transformation group of X consists of local isometries.

zeros of $X \iff$ fixed points of isometries

Chang Wan Kim (2007, J. Math. Kyoto):
M is an even-dimensional compact Finsler manifold of positive flag curvature, then every Killing field has a zero.

Theorem [S. Deng, 2007]

(M, L): connected, forward complete

$V = \{p \in M \mid X(p) = 0\} = \cup V_i$; $\quad V_i$ are connected components.

• each V_i is a totally geodesic closed submanifold of M; codim V_i is even;

• $\forall x \in V_i, y \in V_j, i \neq j$ there is a one-parameter family of geodesics connecting x and y; $\Rightarrow$ x and y are conjugate points.

• M compact; then for the Euler number :

$$\chi(M) = \sum \chi(V_i)$$

Corollary: the flag curvature is non-positive $\Longrightarrow$ V is empty or connected.