The Ruijsenaars self-duality map as a mapping class symlectomorphism

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Integrable systems of Calogero (Moser, Sutherland, Olshanetsky-Perelomov, Ruijsenaars-Schneider) type describe point "particles" moving on the line or on the circle.

These systems are closely connected to soliton theory, for example to the KdV and sine-Gordon models, as well as to Yang-Mills and Chern-Simons field theories, and are located at a crossroad of important areas of mathematics.

They enjoy intriguing "duality relations".

• By definition, two integrable many-body systems are dual to each other if the action variables of system (i) are the particle positions of system (ii), and vice versa. Underlying phase spaces are symplectomorphic.

• A special case of duality is self-duality, where the leading Hamiltonians of the two systems have the same form.

• Equivalently, an integrable many-body Hamiltonian system is called self-dual if it admits a global symplectomorphism that "exchanges" its particle position and action variables. Such symplectomorphism is called the self-duality map and it usually has order 4.

• First example is the self-duality of the rational Calogero system. Interpreted in terms of symplectic reduction by Kazhdan, Kostant and Sternberg (1978).

• Duality was discovered and explored by Ruijsenaars (1988-95) in his direct construction of action-angle variables for Calogero-Sutherland systems and their relativistic deformations. Thus we use terms "Ruijsenaars duality" and "Ruijsenaars self-duality map".

• Our aim is to derive all Ruijsenaars dualities by means of reductions of suitable (finite-dimensional, real) phase spaces.

Duality from reduction: the basic idea

Start with ‘big phase space’, of group theoretic origin, equipped with *two* canonical families of commuting ‘free’ Hamiltonians.

Apply suitable *single* (symplectic) reduction to the big phase space and construct *two* ‘natural’ models of the reduced phase space.

The two families of ‘free’ Hamiltonians turn into interesting **many-body Hamiltonians** and **particle-position variables** in terms of *both* models. Their rôle is *interchanged* in the two models.

The natural symplectomorphism between the two models of the reduced phase space yields the **duality symplectomorphism**.

The above ‘scenario’ was put forward by Gorsky and Nekrasov in the mid-nineties (see e.g. Fock-Gorsky-Nekrasov-Roubtsov [2000]). They focused on local questions working mostly with infinite-dimensional phase spaces and in a complex holomorphic setting. **Global** structure of **real** phase spaces is relevant, and it is one of our main concerns.

The simplest self-dual system: $H_{\text{Cal}}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{x^2}{(q_k-q_j)^2}$

Symplectic reduction: Consider phase space $T^*\mathrm{i}u(n) \simeq \mathrm{i}u(n)\times \mathrm{i}u(n) := \{(Q,P)\}$ with two families of 'free' Hamiltonians $\{\mathrm{tr}\,(Q^k)\}$ and $\{\mathrm{tr}\,(P^k)\}$. Reduce by the adjoint action of $U(n)$ using the moment map constraint

$$[Q,P] = \mu(x) := \mathrm{i}x \sum_{j\neq k} E_{j,k}$$

This yields the self-dual Calogero system (OP [76], KKS [78]):

gauge slice (i): $Q = q := \mathrm{diag}(q_1,\dots,q_n)$, $q_1 > \cdots > q_n$, with $p := \mathrm{diag}(p_1,\dots,p_n)$

$$P = p + \mathrm{i}x \sum_{j\neq k} \frac{E_{jk}}{q_j - q_k} \equiv L_{\text{Cal}}(q,p) \quad \text{Lax matrix,} \quad \mathrm{tr}\,(dP\wedge dQ) = \sum_{k=1}^{n} dp_k \wedge dq_k$$

gauge slice (ii): $P = \hat{p} := \mathrm{diag}(\hat{p}_1,\dots,\hat{p}_n)$, $\hat{p}_1 > \cdots > \hat{p}_n$, with $\hat{q} := \mathrm{diag}(\hat{q}_1,\dots,\hat{q}_n)$

$$Q = -L_{\text{Cal}}(\hat{p},\hat{q}) \quad \text{dual Lax matrix,} \quad \mathrm{tr}\,(dP\wedge dQ) = \sum_{k=1}^{n} d\hat{q}_k \wedge d\hat{p}_k.$$

The alternative gauge slices give two models of the reduced phase space. Their natural symplectomorphism is the self-duality map.

This example motivated Ruijsenaars who also hinted at the possibility of analogous picture in general, and the same example motivated also Gorsky et al.

A ‘dual pair’ of integrable many-body systems

Hyperbolic Sutherland system (1971):

$$H_{\text{hyp}-\text{Suth}}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{x^2}{2}\sum_{j\neq k} \frac{1}{\sinh^2(q_j - q_k)}$$

Basic Poisson brackets: $\{q_i, p_j\} = \delta_{i,j}$, $\quad x$: non-zero, real constant.

Rational Ruijsenaars-Schneider system (1986):

$$H_{\text{rat}-\text{RS}}(\hat{p},\hat{q}) = \sum_{k=1}^{n} \cosh(\hat{q}_k) \prod_{j\neq k} \left[1 + \frac{x^2}{(\hat{p}_k - \hat{p}_j)^2}\right]^{\frac{1}{2}}$$

Poisson brackets: $\{\hat{p}_i, \hat{q}_j\} = \delta_{i,j}$ $\quad$ ($\hat{p}_i$ are RS ‘particle positions’).

Two further dual pairs

Trigonometric Sutherland system

$$H_{\text{trigo-Suth}} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{x^2}{2}\sum_{j\neq k}\frac{1}{\sin^2(q_k - q_j)}$$

and its Ruijsenaars dual

$$\widetilde{H}_{\text{rat-RS}} = \sum_{k=1}^{n}(\cos\hat{q}_k)\prod_{j\neq k}\left[1 - \frac{x^2}{(\hat{p}_k - \hat{p}_j)^2}\right]^{\frac{1}{2}}$$

'Relativistic' deformation (here $c = 1$) of above dual pair:

$$H_{\text{trigo-RS}} = \sum_{k=1}^{n}(\cosh p_k)\prod_{j\neq k}\left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_j)}\right]^{\frac{1}{2}}$$

and the **physically very different** dual system

$$\widehat{H}_{\text{trigo-RS}} = \sum_{k=1}^{n}(\cos\hat{q}_k)\prod_{j\neq k}\left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_k - \hat{p}_j)}\right]^{\frac{1}{2}}$$

Here, naive dual phase spaces need completion [Ruijsenaars, 95].

Other self-dual systems

Hyperbolic Ruijsenaars-Schneider system:

$$H_{\mathrm{hyp-RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j \neq k} \left[1 + \frac{\sinh^2 x}{\sinh^2(q_k - q_j)} \right]^{\frac{1}{2}}$$

Compactified trigonometric RS ($\mathrm{III_b}$) system, locally given by

$$H_{\mathrm{compact-RS}} = \sum_{k=1}^{n} (\cos p_k) \prod_{j \neq k} \left[1 - \frac{\sin^2 x}{\sin^2(q_k - q_j)} \right]^{\frac{1}{2}}$$

These two systems are drastically different.

• **Ruijsenaars duality has "right to exist"** principally because of its usefulness for analyzing the dynamics. This provided the main motivation for Ruijsenaars himself, who used the duality (among others) to analyze the scattering theory of the pertinent systems. Duality has other applications as well, e.g., crucial ingredient in the recent work of Bogomolny, Giraud and Schmit on new integrable random matrix ensembles.

• So far we have successfully elaborated all of Ruijsenaars' examples in the reduction approach except for the hyperbolic RS system.

• Our results connect Ruijsenaars duality to modern developments in symplectic and Poisson geometry, such as Poisson-Lie symmetry and quasi-Hamiltonian manifolds.

• The key symplectomorphism property of the duality map was originally very difficult to prove. It is an automatic consequence in the reduction approach.

Rest of the talk: treat self-duality of compactified trigo RS system

The completed, compact phase space of the local $\mathrm{III_b}$ Hamiltonian $H_x^{\mathrm{loc}} \equiv \sum_{k=1}^{n} (\cos p_k) \prod_{j\neq k} \left[1 - \frac{\sin^2 x}{\sin^2(q_k - q_j)}\right]^{\frac{1}{2}}$ is known to be $\mathbb{C}P(n-1)$.

We reduce the phase space (internally fused quasi-Hamiltonian double)

$SU(n) \times SU(n) = \{(A, B)\}$ by imposing constraint $ABA^{-1}B^{-1} = \mu_0(x)$ using $\mu_0(x) = \mathrm{diag}(e^{2\mathrm{i}x}, ..., e^{2\mathrm{i}x}, e^{2(1-n)\mathrm{i}x})$ and gauge symmetry

$$(A, B) \longrightarrow (gAg^{-1}, gBg^{-1}), \;\; g \in SU(n) \;\text{ with }\; g\mu_0(x)g^{-1} = \mu_0(x).$$

Reduced phase space is moduli space of flat $SU(n)$ connections on the torus with a hole, such that the holonomy around the hole is constrained to conjugacy class of $\mu_0(x)$. The matrices A and B are the holonomies along the standard cycles on the torus, and their invariant functions generate two Abelian Poisson algebras.

For geometric reasons, the mapping class group of the “one-holed torus” – $SL(2, \mathbb{Z})$ – acts symplectically on the reduced phase space.

• First result: We have shown that the reduced phase space is $\mathbb{C}P(n-1)$ and "spectral functions" of A and B descend to global particle position and action variables of the compactified RS system.

• Second result: We have proved that the usual "duality generator" $S \in SL(2,\mathbb{Z})$ induces Ruijsenaars' self-duality symplectomorphism of the compactified RS system.

Thus we have proved conjectures and "globalized" local results of Gorsky-Nekrasov [95] and Fock-Gorsky-Nekrasov-Roubtsov [2000].

The precise global treatment relies on quasi-Hamiltonian geometry.

Quasi-Hamiltonian geometry [Alekseev-Malkin-Meinrenken 1998]

Let G be a compact Lie group. The G-manifold M equipped with the invariant 2-form ω is called quasi-Hamiltonian if there exists a (moment) map $\mu : M \to G$ such that:

$$d\omega = -\frac{1}{12}\langle \mu^{-1}d\mu, [\mu^{-1}d\mu, \mu^{-1}d\mu]\rangle;$$

$$\omega(\zeta_M, \cdot) = \frac{1}{2}\langle \mu^{-1}d\mu + d\mu\mu^{-1}, \zeta\rangle, \quad \forall \zeta \in \mathrm{Lie}(G);$$

$$\mathrm{Ker}(\omega_m) = \{\zeta_M(m) \,|\, \zeta \in \mathrm{Ker}(\mathrm{Ad}_{\mu(m)} + \mathrm{Id}_{\mathrm{Lie}(G)})\}, \quad \forall m \in M;$$

$$\mu(\Psi_g(m)) = g(\mu(m))g^{-1}, \qquad \forall m \in M, \quad \forall g \in G.$$

Here $\langle \cdot, \cdot \rangle$ denotes an invariant scalar product on Lie(G), ζ_M is the vector field on M that corresponds to $\zeta \in$ Lie(G), and $\Psi_g : M \to M$ is the action of $g \in G$.

QUASI-HAMILTONIAN DYNAMICAL SYSTEMS

The axioms of quasi-Hamiltonian geometry imply that for each G-invariant function α there exists a unique G-invariant ("quasi-Hamiltonian") vector field v_α on M verifying

$$\omega(v_\alpha, \cdot) = d\alpha \quad \text{and} \quad \mathcal{L}_{v_\alpha}\mu = 0. \qquad (*)$$

Moreover, the function

$$\{\alpha, \beta\} := \omega(v_\alpha, v_\beta) \qquad (**)$$

is G-invariant and thus one obtains a Poisson bracket on the space of G-invariant functions on M. This yields an honest Poisson structure on M/G, if M/G is an honest manifold.

The G-invariant functions on M are called Hamiltonians and, via $(*)$, they induce G-invariant evolution flows on M that descend to the symplectic leaves of M/G.

Thus one can work with G-invariant functions on quasi-Hamiltonian manifolds in much the same way as with arbitrary functions on symplectic manifolds.

QUASI-HAMILTONIAN REDUCTION

With $h \in C^\infty(M)^G$, consider a quasi-Hamiltonian dynamical system (M, G, ω, μ, h), an element $\mu_0 \in G$, the isotropy subgroup $G_0 < G$ of μ_0 with respect to the adjoint action and the "constraint surface"

$$C_{\mu_0} = \mu^{-1}(\mu_0) = \{m \in M \,|\, \mu(m) = \mu_0\}.$$

We say that μ_0 is strongly regular if C_{μ_0} is an embedded submanifold of M and the quotient $P(\mu_0) \equiv C_{\mu_0}/G_0$ is a manifold such that the canonical projection $p : C_{\mu_0} \to C_{\mu_0}/G_0$ is a smooth submersion. For strongly regular μ_0 there is a **symplectic** form $\hat{\omega}$ and a Hamiltonian $\hat{h}$ on $P(\mu_0)$ uniquely defined by

$$p^*\hat{\omega} = \iota^*\omega, \quad p^*\hat{h} = \iota^*h$$

with tautological embedding $\iota : C_{\mu_0} \to M$.

Hamiltonian vector field and flow defined by $\hat{h}$ on **reduced phase space** $P(\mu_0)$ can be obtained by first restricting the quasi-Hamiltonian vector field v_h and its flow to the constraint surface C_{μ_0} and then applying projection p. /$P(\mu_0) \simeq \mu^{-1}(G\mu_0)/G$ is stratified symplectic space in general./

EXAMPLE: INTERNALLY FUSED DOUBLE

This quasi-Hamiltonian manifold (D, G, ω, μ) is provided by direct product

$$D := G \times G = \{(A, B) \,|\, A, B \in G\}.$$

The group G acts on D by componentwise conjugation

$$\Psi_g(A, B) := (gAg^{-1}, gBg^{-1}), \qquad \forall g \in G.$$

The 2-form ω on D reads

$$\omega := \frac{1}{2}\langle A^{-1}dA \overset{\wedge}{,} dBB^{-1}\rangle + \frac{1}{2}\langle dAA^{-1} \overset{\wedge}{,} B^{-1}dB\rangle$$

$$-\frac{1}{2}\langle (AB)^{-1}d(AB) \overset{\wedge}{,} (BA)^{-1}d(BA)\rangle,$$

and the G-valued moment map μ is defined by

$$\mu(A, B) = ABA^{-1}B^{-1}.$$

We take $G := SU(n)$ and invariant scalar product

$$\langle \eta, \zeta \rangle := -\frac{1}{2}\mathrm{tr}\,(\eta\zeta), \qquad \forall \eta, \zeta \in su(n) \equiv \mathrm{Lie}(SU(n)).$$

Mapping class group action on D and on $P(\mu_0)$

Consider the (orientation-preserving) mapping class group of the "one-holed torus" Σ,

$$\mathrm{MCG}^+(\Sigma) \equiv \pi_0(\mathrm{Diff}^+(\Sigma)) \simeq SL(2,\mathbb{Z}),$$

which is generated by two elements S and T subject to

$$S^2 = (ST)^3, \quad S^4 = 1.$$

As concrete matrices, one may take

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

In association to S and T, define diffeomorphisms S_D and T_D of D:

$$S_D(A,B) := \Psi_B(B^{-1},A) = (B^{-1}, BAB^{-1}), \quad T_D(A,B) := (AB,B)$$

In fact, S_D and T_D are automorphisms of the double D:

$$S_D^*\omega = \omega, \quad S_D \circ \Psi_g = \Psi_g \circ S_D, \quad \mu \circ S_D = \mu, \quad \text{and similar for} \quad T_D.$$

Moreover, S_D and T_D satisfy

$$S_D^2 = (S_D \circ T_D)^3, \quad S_D^4 = Q,$$

where Q is the following "universal central automorphism" of D:

$$Q(A, B) := \Psi_{\mu(A,B)^{-1}}(A, B).$$

S_D and T_D descend to maps S_P and T_P on any reduced phase space $P(\mu_0)$, and **these maps generate $SL(2, \mathbb{Z})$ action on $P(\mu_0)$.** Indeed, Q descends to the trivial identity map id_P on $P(\mu_0)$, and therefore

$$S_P^2 = (S_P \circ T_P)^3, \quad S_P^4 = \mathrm{id}_P.$$

Resulting $SL(2, \mathbb{Z})$ action preserves (stratified) symplectic structure on $P(\mu_0)$.

This is a concrete description of the standard mapping class group action on the moduli space $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

“Free” Hamiltonians on the double and their reductions

For any $\mathcal{H} \in C^\infty(G)^G$, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the invariant functions on D given by $\mathcal{H}_1(A,B) := \mathcal{H}(A)$ and $\mathcal{H}_2(A,B) := \mathcal{H}(B)$. Then $\{\mathcal{H}_1\}$ and $\{\mathcal{H}_2\}$ form two Abelian Poisson algebras on D. One can easily write down the corresponding quasi-Hamiltonian flows on D.

By reduction, one obtains two Abelian Poisson algebras on each reduced phase space $P(\mu_0)$:

$$\mathcal{C}^a := \{\hat{\mathcal{H}}_1 \,|\, \mathcal{H} \in C^\infty(G)^G\}, \quad \mathcal{C}^b := \{\hat{\mathcal{H}}_2 \,|\, \mathcal{H} \in C^\infty(G)^G\}.$$

These Abelian algebras are interchanged under the action of S_P.

More exactly, for $\mathcal{H} \in C^\infty(G)^G$, define $\mathcal{H}^\sharp \in C^\infty(G)^G$ by $\mathcal{H}^\sharp(g) := \mathcal{H}(g^{-1})$. Then

$$\hat{\mathcal{H}}_2 \circ S_P = \hat{\mathcal{H}}_1 \quad \text{and} \quad \hat{\mathcal{H}}_1 \circ S_P = \hat{\mathcal{H}}_2^\sharp, \quad \forall \mathcal{H} \in C^\infty(G)^G.$$

Some more preparations

For any $-\frac{\pi}{n} < x < \frac{\pi}{n}$ introduce the closed polytope

$$\mathcal{P}_x := \Big\{(\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1} \,\Big|\, \xi_j \geq |x|,\ \ j = 1, \ldots, n-1,\ \ \sum_{j=1}^{n-1} \xi_j \leq \pi - |x|\Big\}$$

Prepare the $n \times n$ matrices

$$H_k := E_{k,k} - E_{k+1,k+1}, \quad \lambda_k := \sum_{j=1}^{k} E_{j,j} - \frac{k}{n}\mathbf{1}_n, \quad k = 1, \ldots, n-1.$$

For $\xi \in \mathcal{P}_0$ and $\tau = (\tau_1, \ldots, \tau_{n-1}) = (e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_{n-1}}) \in \mathbb{T}_{n-1}$ define

$$\delta(\xi) := \exp\Big(-2\mathrm{i} \sum_{k=1}^{n-1} \xi_k \lambda_k\Big), \quad \Theta(\tau) := \exp\Big(-\mathrm{i} \sum_{k=1}^{n-1} \theta_k H_k\Big).$$

Since any $g \in G = SU(n)$ is conjugate to $\delta(\xi)$ for unique $\xi \in \mathcal{P}_0$, can specify conjugation invariant function Ξ_k on G by

$$\Xi_k(\delta(\xi)) := \xi_k, \quad \forall \xi \in \mathcal{P}_0, \quad k = 1, \ldots, n-1.$$

The "**spectral function**" Ξ_k is continuous on G and its restriction to G_{reg} belongs to $C^\infty(G_{\mathrm{reg}})^G$.

Local $\mathrm{III_b}$ system: $\delta_j = e^{\mathrm{i}2q_j}$ $(j = 1, ..., n)$ are particle positions and their conjugate momenta p_j encode $\Theta_j = e^{-\mathrm{i}p_j}$. To keep $H_x^{\mathrm{loc}} \equiv \sum_{j=1}^n (\cos p_j) \prod_{k\neq j}^n \left[1 - \frac{\sin^2 x}{\sin^2(q_j - q_k)}\right]^{\frac{1}{2}}$ real on a connected open domain, $|x| < |q_j - q_k| < \pi - |x|$ must hold for all $j \neq k$ and the coupling constant $x \neq 0$ must satisfy $0 < |x| < \pi/n$.

Impose center of mass condition $\prod_{j=1}^n \delta_j = \prod_{j=1}^n \Theta_j = 1$. Take local phase space

$$M_x^{\mathrm{loc}} \equiv \mathcal{P}_x^0 \times \mathbb{T}_{n-1},$$

using the parametrization $\mathcal{P}_x^0 \times \mathbb{T}_{n-1} \ni (\xi, \tau) \mapsto (\delta(\xi), \Theta(\tau))$.

Symplectic form and $SU(n)$-valued local Lax matrix are

$$\Omega^{\mathrm{loc}} := \frac{1}{2}\mathrm{tr}\left(\delta^{-1}d\delta \wedge \Theta^{-1}d\Theta\right) = \mathrm{i}\sum_{k=1}^{n-1} d\xi_k \wedge \tau_k^{-1}d\tau_k = \sum_{k=1}^{n-1} d\theta_k \wedge d\xi_k.$$

$$L_{\mathrm{loc}}^x(\xi,\tau)_{jl} := \frac{e^{\mathrm{i}x} - e^{-\mathrm{i}x}}{e^{\mathrm{i}x}\delta_j(\xi)\delta_l(\xi)^{-1} - e^{-\mathrm{i}x}} W_j(\xi,x)W_l(\xi,-x)\Theta_l(\tau)\Delta_l(\tau)\Delta_j(\tau)^{-1}$$

with the positive functions

$$W_j(\xi,x) := \prod_{k\neq j}^n \left[\frac{e^{\mathrm{i}x}\delta_j(\xi) - e^{-\mathrm{i}x}\delta_k(\xi)}{\delta_j(\xi) - \delta_k(\xi)}\right]^{\frac{1}{2}} \quad \text{and} \quad \Delta(\tau) := \mathrm{diag}(\tau_1, ..., \tau_{n-1}, 1)$$

$H_x^{\mathrm{loc}} = \mathrm{Re}\,\mathrm{tr}(L_{\mathrm{loc}}^x)$ and the spectral invariants of L_{loc}^x Poisson commute.

To complete $\mathrm{III_b}$ system, consider $(\mathbb{C}P(n-1), \chi_0\omega_{\mathrm{FS}})$ using $\chi_0 \equiv \pi - n|x|$ and

$$\mathbb{C}P(n-1) = S^{2n-1}_{\chi_0}/U(1), \qquad S^{2n-1}_{\chi_0} = \left\{(u_1, ..., u_n) \in \mathbb{C}^n \mid \sum_{k=1}^{n} |u_k|^2 = \chi_0\right\}$$

With $\pi_{\chi_0} : S^{2n-1}_{\chi_0} \to \mathbb{C}P(n-1)$, define **symplectic embedding** $\mathcal{E} : M^{\mathrm{loc}}_x \to \mathbb{C}P(n-1)$

$$\mathcal{E}(\xi, \tau) := \pi_{\chi_0}(\tau_1\sqrt{\xi_1 - |x|}, ..., \tau_{n-1}\sqrt{\xi_{n-1} - |x|}, \sqrt{\xi_n - |x|}), \quad \xi_n := \pi - \sum_{k=1}^{n-1} \xi_k$$

$(M^{\mathrm{loc}}_x, \Omega^{\mathrm{loc}})$ is mapped onto **dense open** submanifold $\mathbb{C}P(n-1)_0 \subset \mathbb{C}P(n-1)$.
L^x_{loc} **extends smoothly** and yields global Lax matrix $L^x \in C^\infty(\mathbb{C}P(n-1), SU(n))$.

Compactified $\mathrm{III_b}$ system, $(\mathbb{C}P(n-1), \chi_0\omega_{\mathrm{FS}}, L^x)$, has two distinguished Abelian Poisson algebras. First spanned by "**global particle positions**" $\mathcal{J}_k$:

$$\mathcal{J}_k \circ \pi_{\chi_0}(u) := |u_k|^2 + |x|, \qquad k = 1, ..., n-1.$$

The $\mathcal{J}_k$ satisfy $\mathcal{J}_k(\mathcal{E}(\xi, \tau)) = \xi_k$ and form toric moment map

$$\mathcal{J} := (\mathcal{J}_1, ..., \mathcal{J}_{n-1}) : \mathbb{C}P(n-1) \to \mathbb{R}^{n-1}$$

generating rotational action of $\mathbb{T}_{n-1}$ on $\mathbb{C}P(n-1)$. Its image is the polytope $\mathcal{P}_x$.

Second Abelian algebra is spanned by **action variables** furnished by $\mathcal{I}_k := \Xi_k \circ L^x$.

To present our main results, define the spectral Hamiltonians α_k and β_k on D by

$$\alpha_k(A,B) := \Xi_k(A) \quad \text{and} \quad \beta_k(A,B) := \Xi_k(B)$$

They descend to "reduced spectral Hamiltonians" $\hat{\alpha}_k$ and $\hat{\beta}_k$ on any reduced phase space $P(\mu_0)$. Under the $SL(2,\mathbb{Z})$ generator S_P:

$$\hat{\beta}_k \circ S_P = \hat{\alpha}_k \quad \text{and} \quad \hat{\alpha}_k \circ S_P = \hat{\beta}_{n-k}, \quad \forall k = 1, ..., n-1.$$

Theorem 1. *For the particular moment map value*

$$\mu_0 = \text{diag}(e^{2ix}, ..., e^{2ix}, e^{2(1-n)ix}), \qquad 0 < |x| < \pi/n,$$

the "constraint surface" $\mu^{-1}(\mu_0)$ lies in $G_{\text{reg}} \times G_{\text{reg}}$. The reduced phase space $(P(\mu_0), \hat{\omega})$ is smooth and is symplectomorphic to $(\mathbb{C}P(n-1), \chi_0\omega_{\text{FS}})$. The maps

$$\hat{\alpha} := (\hat{\alpha}_1, ..., \hat{\alpha}_{n-1}) : P(\mu_0) \to \mathbb{R}^{n-1} \quad \textit{and} \quad \hat{\beta} : (\hat{\beta}_1, ..., \hat{\beta}_{n-1}) : P(\mu_0) \to \mathbb{R}^{n-1}$$

are toric moment maps generating two effective Hamiltonian actions of $\mathbb{T}_{n-1}$ on $(P(\mu_0), \hat{\omega})$. The images of both $\hat{\alpha}$ and $\hat{\beta}$ yield the polytope $\mathcal{P}_x$. There exists a symplectomorphism

$$f_\beta : \mathbb{C}P(n-1) \to P(\mu_0)$$

that satisfies

$$\hat{\beta}_k \circ f_\beta = \mathcal{J}_k \quad \textit{and} \quad \hat{\alpha}_k \circ f_\beta = \mathcal{I}_k, \qquad \forall k = 1, ..., n-1.$$

Therefore, f_β converts the toric moment maps $\hat{\beta}$ and $\hat{\alpha}$ respectively into the particle positions $\mathcal{J}$ and action variables $\mathcal{I}$ of the compactified RS III_b system.

Corollary 1. *The symplectomorphisms* $f_\beta^{-1} \circ S_P \circ f_\beta$ *and* $f_\beta^{-1} \circ T_P \circ f_\beta$ *generate an* $SL(2,\mathbb{Z})$ *action on the compactified* $\mathrm{III_b}$ *phase space* $(\mathbb{C}P(n-1), \chi_0 \omega_{\mathrm{FS}})$. *The mapping class "duality symplectomorphism"*

$$\mathfrak{S} := f_\beta^{-1} \circ S_P \circ f_\beta$$

acts by exchanging particle positions $\mathcal{J}_k$ *with action variables* $\mathcal{I}_k$ *according to*

$$\mathcal{J}_k \circ \mathfrak{S} = \mathcal{I}_k, \quad \textit{and} \quad \mathcal{I}_k \circ \mathfrak{S} = \mathcal{J}_{n-k}, \qquad \forall k = 1, ..., n-1.$$

We have **constructed** the map f_β:

Using $v_j(\xi, x) := \left[\frac{\sin x}{\sin nx}\right]^{\frac{1}{2}} W_j(\xi, x)$, introduce $g_x(\xi) \in U(n)$ for each $\xi \in \mathcal{P}_x^0$ by

$$g_x(\xi)_{jn} := -g_x(\xi)_{nj} := v_j(\xi, x), \quad \forall j = 1, ..., n-1, \quad g_x(\xi)_{nn} := v_n(\xi, x),$$

$$g_x(\xi)_{jl} := \delta_{jl} - \frac{v_j(\xi, x) v_l(\xi, x)}{1 + v_n(\xi, x)}, \quad \forall j, l = 1, ..., n-1.$$

Theorem 2. *The map* $f_0 : \mathbb{C}P(n-1)_0 \to P(\mu_0)$ *defined by*

$$(f_0 \circ \mathcal{E})(\xi, \tau) := p \circ \Psi_{g_x(\xi)^{-1}} \circ \Psi_{\Delta(\tau)} (L^x_{\mathrm{loc}}(\xi, \tau), \delta(\xi))$$

is a diffeomorphism from $\mathbb{C}P(n-1)_0$ *onto a dense open submanifold of* $P(\mu_0)$. *This map is* **symplectic**, $f_0^* \hat{\omega} = \chi_0 \omega_{\mathrm{FS}}$, *and it* **extends** *to a global diffeomorphism* $f_\beta : \mathbb{C}P(n-1) \to P(\mu_0)$, *which satisfies* $\hat{\beta}_k \circ f_\beta = \mathcal{J}_k$ *and* $\hat{\alpha}_k \circ f_\beta = \mathcal{I}_k$.

/We described f_β even more explicitly using n coordinate patches on $\mathbb{C}P(n-1)$./

CONCLUSION

Mapping class symplectomorphism $\mathfrak{S}$ qualifies as self-duality map. In fact, $\mathfrak{S}$ reproduces the self-duality symplectomorphism of the compactified $\mathrm{III_b}$ system constructed originally by a very different (non-geometric, direct) method by Ruijsenaars [1995].

Remark: A second parameter can be introduced into the system by scaling the 2-form of the double D. This becomes important quantum mechanically.

We have also applied reduction methods to obtain other many-body systems together with geometric interpretation of their duality relations.

Principal advantage of the reduction approach:
Once the correct starting point is 'guessed', completion of local phase spaces and duality symplectomorphisms result automatically. This approach also links integrable many-body systems and their duality to several interesting subjects.

An important open problem: How to obtain the hyperbolic RS system?