Small complete caps in Galois spaces from bicovering arcs

In an (affine or projective) space of dimension $N \geq 2$ over the finite field with q elements $\mathbb{F}_q$, a k-cap is a set of k points no three of which are collinear. A k-cap is said to be complete if its secants cover all the points of the space. In the plane, that is for $N = 2$, k-caps are also called k-arcs. The general theory of k-caps was developed in 1960's by the pioneering work of Segre. Ever since, k-caps and their generalizations, especially (k, d)-arcs, saturating sets and k-arcs in higher dimensions, have played an important role in Finite Geometry. All these objects are relevant not only in Finite Geometry but also in Coding Theory, being the geometrical counterpart of distinguished types of error-correcting and covering linear codes.

In this direction, an important issue is to ask for explicit constructions of complete k-caps in higher dimensional spaces. Since the theory of plane k-arcs is well developed and quite rich of constructions, the natural idea is to try to using some kind of lifting methods for plane k-arcs to obtain complete caps in higher dimension.

For this purpose, bicovering arcs in affine planes have recently emerged as a potential powerful tool. Here, a k-arc A in the affine plane $AG(2, q)$, q odd, is said to be bicovering if its completeness holds in a stronger sense: it is required that every point P off A is covered by at least two secants of A, in such a way that P is external to the segment cut out by one of the secants but it is internal when the other secant is considered. In other words, and in analogy with Segre's terminology, a complete k-arc A is bicovering if no point off A is either regular or pseudo-regular.

Complete caps from bicovering arcs can be obtained via the product method for caps: the cartesian product of a bicovering k-arc A in $AG(2, q)$ and the subset of $AG(N - 2, q)$ arising from the blow-up of a parabola of $AG(2, q^{(N-2)/2})$ is a complete cap of size $kq^{(N-2)/2}$ in $AG(N, q)$, provided that $N \equiv 0 \pmod 4$.

To establish whether a complete arc is bicovering can be a difficult task. In the (simplest) case where the arc A consists of the affine points of an irreducible conic, if q is large enough then A bicovers all the points off A with at most one exception. This follows from previous results by Segre.

In this talk we are going to presente some recent constructions of small bicovering arcs arising from both singular and elliptic cubic curves defined over $\mathbb{F}_q$.