How to sew in practice?

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Joint work with Ryan Trelford (Univ. of Calgary)

The naïve method

Algorithm:

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1. Get married!
2. Stop.

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2. Stop.

Several disadvantages:

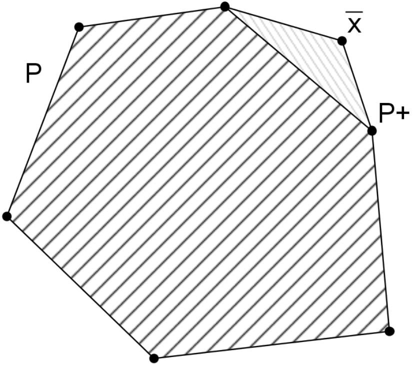
- might be very time consuming
- unreliable
- just doubles the workload for most women

Introduction

The rough idea of sewing is the following: given a polytope P with as many faces as possible, we would like to constuct a new polytope P^+ such that

$$\mathrm{vert}\, P^+ = \mathrm{vert}\, P \cup \{\bar{x}\},$$

with $\bar{x} \notin P$, and P^+ has still as many faces as possible.



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We shall work in the d-dimensional Euclidean space $\mathbb{E}^d$. We will usually assume that d is even, in this case we write $d = 2m$.

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Definition

A d-polytope P is **neighbourly** if every $\lfloor \frac{d}{2} \rfloor$ vertices determine a face of P.

- This is the strongest meaningful definition, since if every $\lfloor \frac{d}{2} \rfloor + 1$ vertices determine a face of P, then P is a simplex.
- All 2- and 3-polytopes are neighbourly.
- If d is even, then neighbourly polytopes are simplicial.

Are there neighbourly polytopes?

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Cyclic polytopes

Consider the *moment curve*

$$m(t) = (t, t^2, t^3, \ldots, t^d)$$

in $\mathbb{E}^d$, and choose n points $v_1 = m(t_1), v_2 = m(t_2), \ldots, v_n = m(t_n)$ $(t_1 < t_2 < \ldots < t_n)$ on it. The polytope

$$C(n, d) = [v_1, v_2, \ldots, v_n]$$

is called the d-dimensional cyclic polytope with n vertices. The combinatorial structure of this polytope is independent of the points chosen, that is any two of them are combinatorially equivalent. Furthermore, if a polytope P is combinatorially equivalent to $C(n, d)$ for some n and d, then P is also called a cyclic polytope.

About the facelattice of cyclic polytopes

Gale's evenness condition

Let $n > 2m+2$, and let $C = C(n, 2m)$ be a cyclic polytope as above, and set $v_{n+1} = v_1$. Then F is a facet of C if, and only if, there exist $i_1 < i_2 < \ldots < i_m$ such that

$$\left| \bigcup_{k=1}^{m} \{v_{i_k}, v_{i_k+1}\} \right| = 2m \quad \text{and} \quad F = \left[\bigcup_{k=1}^{m} \{v_{i_k}, v_{i_k+1}\} \right].$$

Remark: Every set V of vertices determines a face if, and only if, V is a subset of a facet, thus Gale's evenness condition gives us all the information about the face lattice of $C(n, 2m)$.

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Remark: Every set V of vertices determines a face if, and only if, V is a subset of a facet, thus Gale's evenness condition gives us all the information about the face lattice of $C(n, 2m)$.
The proof of the statement depends on the fact, that if a hyperplane H intersects $m(t)$ in exactly $2m$ points, then every intersection point is a cutpoint (the curve "passes through" the hyperplane). The neighbourlyness of $C(n, 2m)$ readily follows from the remark. $o/\square$

History

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However, it remained a challenging task to construct neighbourly polytopes other than the cyclic ones.

In 1982 Shemer introduced the idea of sewing which finally allowed one to construct infinite families of neighbourly polytopes (but not all neighbourly polytopes can be obtained by sewing).

Quotient polytopes

Definition

Let P be a d-polytope, x_0 be a vertex of P, and let H be a hyperplane that separates x_0 from the other vertices of P. Then $H \cap P$ is a $(d-1)$-polytope called the vertex figure of P at x_0.

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- $P/\{x_0\} = H \cap P$.
- $F \in \mathcal{F}(P/\{x_0\})$ if and only if $[F, x_0] \in \mathcal{F}(P)$ (P simplicial).
  - $(\mathcal{F}(P/\{x_0\}), \subseteq)$ is a sublattice of $(\mathcal{F}(P), \subseteq)$.
- Any two vertex figures of P at x_0 are combinatorially equivalent.

Quotient polytopes

Lemma

Let P be a d-polytope, and G be a face of P. Consider the sublattice $\hat{L}$ of the face lattice of P that is generated by G. Then there exists a polytope $\hat{P}$ with the face lattice given by $\hat{L}$. Furthermore, $\hat{P}$ is unique up to combinatorial equivalence.

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We can consider quotient polytopes as iterated vertex figures. $o/\Box$

Universal faces

Definition (Universal face)

Let Q be a neighbourly $2m$-polytope, and let Φ be a k-face of Q, $0 \leq k \leq d-1$. We call Φ a **universal** k-face if $[\Phi, A]$ is a face of Q for every $A \subseteq (\mathcal{V}(Q) \setminus \mathcal{V}(\Phi))$ where

$$|A| \leq \left\lfloor \frac{1}{2}(2m - |\mathcal{V}(\Phi)|) \right\rfloor .$$

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Lemma

Let Q be a neighbourly $2m$-polytope, and Φ, Ψ be faces of Q with $\Phi \subset \Psi$, and assume that Φ is universal in Q. Then Ψ is universal in Q if and only if Ψ/Φ is universal in Q/Φ.

Universal faces of cyclic polytopes

Extended Gale's Evenness Condition

Let $n > 2m+2$ and $C = C(n, 2m)$ be a cyclic polytope with the vertex array $v_1 = m(t_1), \ldots, v_n = m(t_n)$, $t_1 < t_2 < \ldots < t_n$, and $v_{n+1} = v_1$. Then $U \in \mathcal{U}_{2j-1}(C)$, for some $j = 1, \ldots, m$ if, and only if, there exist $i_1 < i_2 < \ldots < i_j$ such that

$$\left| \bigcup_{k=1}^{j} \{v_{i_k}, v_{i_k+1}\} \right| = 2j \qquad \text{and} \qquad U = \left[\bigcup_{k=1}^{j} \{v_{i_k}, v_{i_k+1}\} \right].$$

In other words, odd dimensional universal faces are disjoint unions of universal edges.

Note that, by the previous lemma disjoint unions of universal edges are always universal faces, however the converse is not true in general.

Universal tower

Definition

Let Q be a neighbourly $2m$-polytope, and let $\mathcal{T} = \{\Phi_j\}_{j=1}^m$, where $\Phi_j \in \mathcal{U}_{2j-1}(Q)$ for $j = 1, \ldots, m$ such that $\Phi_1 \subset \Phi_2 \subset \ldots \subset \Phi_m$. We say that $\mathcal{T}$ is a **universal tower** of Q.

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We write

$$\Phi_1 = [x_1, y_1],$$

and for $j = 2, \ldots, m$,

$$\Phi_j = [x_j, y_j, \Phi_{j-1}].$$

When necessary, we set $\Phi_0 = \emptyset$. $o/\square$

The idea of sewing

Let Q be a $2m$-dimensional polytope, and $\mathcal{T}$ a tower in Q. For every facet F of Q let i_F be the maximal integer with $\Phi_{i_F} \subseteq F$. There exists a point $\bar{x}(\mathcal{T}) = \bar{x}$ such that $\bar{x}$ sees F if and only if i_F is odd.

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Note that not every neighbourly polytope has a universal tower, but we can start with a cyclic polytope, and it can be shown that a sewn polytope always has a universal tower, so the sewing process never stops.

Beneath and beyond

Φ_0	beneath	Category I
Φ_1	beyond	Category II
Φ_2	beneath	Category III
$\vdots$		

Problem

How to sew in practise?

A Combinatorial Equivalence

Theorem (R. Trelford, V. V. (2011+))

Let Q be a neighbourly $2m$-polytope, $\mathcal{T} = \{\Phi_j\}_{j=1}^m$ be a universal tower in Q. Let $Q^+ = [Q, \bar{x}]$ be sewn through $\mathcal{T}$, and $(Q/\Phi_1)^+ = [Q/\Phi_1, \bar{z}^*]$ sewn through $\mathcal{T}/\Phi_1$. Then

$$(Q/\Phi_1)^+ \cong Q^+/[x_1, \bar{x}] \cong Q^+/[y_1, \bar{x}].$$

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The bijection of the vertices is given by

- $v^* \mapsto v^{**}$ *if* $v \in \mathcal{V}(Q) \setminus \{x_1, y_1\}$
- $\bar{z}^* \mapsto y_1^{**}$ *(or* $\bar{z}^* \mapsto x_1^{**}$*)*

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Lemma

Any facet of Q^+ that is not a facet of Q contains $\bar{x}$ and at least one of x_1 and y_1.

Generalization

Theorem (R. Trelford, V. V., 2011+)

With the notation above;

$$(P/\Phi_i)^+ \overset{\varphi}{\cong} P^+/[\Phi_{i-1}, x_i, \bar{x}] \ \left(\cong P^+/[\Phi_{i-1}, y_i, \bar{x}]\right) \qquad \text{for } 1 \le i \le m,$$

with the bijection φ of the vertices given by $v^* \mapsto v^{**}$ if $v \in \mathcal{V}(P) \setminus \mathcal{V}(\Phi_i)$, and $\bar{y}^* \mapsto y_i^{**}$. For $v \in \mathcal{V}(P^+) \setminus (\mathcal{V}(\Phi_{i-1}) \cup \{x_i, \bar{x}\})$, v^{**} denotes the corresponding vertex of $P^+/[\Phi_{i-1}, x_i, \bar{x}]$.

Keeping track of universal faces

Theorem. (R. Trelford, V. V., 2011+)

Let $U \in \mathcal{U}_{2k-1}(P)$, $0 \leq i < k < m$, $\Phi_i \subseteq U$ and $x_{i+1} \notin U$. Then $U \in \mathcal{U}_{2k-1}(P^+)$ if, and only if, i is even and $[U, \bar{x}, x_{i+1}] \in \mathcal{U}_{2k+1}(P^+)$.

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Based on this theroem, the sewing algorithm can be extended such that given the list of all odd dimensional universal faces of P, we obtain the same information about P^+. The running time is a log-factor worse than the theoretical best possible.

Dimension	$(P/\Phi_{m-1})^+$		$(P/\Phi_{m-2})^+$		$\cdots$		$(P/\Phi_2)^+$		$(P/\Phi_1)^+$		P^+
$2m-1$											$\frac{m^2-m+2}{2}$
										$\nearrow$	$\downarrow$
$2m-3$									$\frac{m^2-3m+4}{2}$		$\frac{m^2-m+4}{2}$
								$\nearrow$	$\downarrow$	$\nearrow$	$\downarrow$
$\vdots$							$\vdots$		$\vdots$		$\vdots$
							$\downarrow$	$\nearrow$	$\downarrow$	$\nearrow$	$\downarrow$
5							$\frac{m^2-3m-2}{2}$		$\frac{m^2-m-4}{2}$		$\frac{m^2+m-4}{2}$
				$\nearrow$	$\cdots$	$\nearrow$	$\downarrow$	$\nearrow$	$\downarrow$	$\nearrow$	$\downarrow$
3			2		$\cdots$		$\frac{m^2-3m}{2}$		$\frac{m^2-m-2}{2}$		$\frac{m^2+m-2}{2}$
		$\nearrow$	$\downarrow$	$\nearrow$	$\cdots$	$\nearrow$	$\downarrow$	$\nearrow$	$\downarrow$	$\nearrow$	$\downarrow$
1	1		3		$\cdots$		$\frac{m^2-3m+2}{2}$		$\frac{m^2-m}{2}$		$\frac{m^2+m}{2}$

Table: The order of the steps in Algorithm 2

Open questions

- T. Bisztriczky (2000) generalized the sewing method for odd dimensional polytopes. It is very natural to ask wheter the results extend into odd dimensions.

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- Find a new construction method to obtain neighbourly polytopes.
- In particular examine the family of neighbourly polytopes that are subpolytopes of totally sewn polytopes.

Thank you very much for your attention!