Asymptotic behavior of supercritical multi-type continuous state and continuous time branching processes with immigration

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- Multi-type CBI processes (MCBI processes)
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Galton–Watson branching process with immigration

GWI process:

$$\zeta_k = \sum_{j=1}^{\zeta_{k-1}} \xi_{k,j} + \varepsilon_k, \qquad k \in \mathbb{N} := \{1, 2, \ldots\},$$

$\{\xi_{k,j}, \, \varepsilon_k : k, j \in \mathbb{N}\}$ independent rv's with values in $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$

$\{\xi_{k,j} : k, j \in \mathbb{N}\}$ identically distributed

$\{\varepsilon_k : k \in \mathbb{N}\}$ identically distributed

Possible scaling limits: CBI processes (Kawazu & Watanabe, 1971; Li, 2006)

$\forall n \in \mathbb{N}$, let $(\zeta_k^{(n)})_{k\in\mathbb{Z}_+}$ be a GWI process, and $\gamma_n \in \mathbb{R}_{++}$ with $\gamma_n \uparrow \infty$. Under certain conditions, $(n^{-1}\zeta_{\lfloor \gamma_n t\rfloor}^{(n)})_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (X_t)_{t\in\mathbb{R}_+}$ as $n \to \infty$, where $(X_t)_{t\in\mathbb{R}_+}$ is a conservative time-homogeneous Markov process with state space $\mathbb{R}_+$ and with infinitesimal generator

$$(\mathcal{A}f)(x) = (bx+\beta)f'(x) + cxf''(x) + \int_0^\infty \big[f(x+z) - f(x)\big]\,\nu(\mathrm{d}z) + x\int_0^\infty \big[f(x+z) - f(x) - f'(x)(1\wedge z)\big]\,\mu(\mathrm{d}z)$$

for $f \in \mathbb{C}_\mathrm{c}^2(\mathbb{R}_+,\mathbb{R})$ and $x \in \mathbb{R}_+$, where $b \in \mathbb{R}$, $\beta, c \in \mathbb{R}_+$, and ν, μ are Borel measures on $(0,\infty)$ with $\int_0^\infty (1\wedge z)\,\nu(\mathrm{d}z) < \infty$ and $\int_0^\infty (z\wedge z^2)\,\mu(\mathrm{d}z) < \infty$.

The Markov process $(X_t)_{t\in\mathbb{R}_+}$ is called a CBI process with parameter vector (b, c, μ, β, ν).

SDE of a single-type CBI process (Dawson & Li, 2006)

If $\int_1^\infty z\,\nu(\mathrm{d}z) < \infty$ then there is a pathwise unique non-negative strong solution to SDE

$$X_t = X_0 + \int_0^t (\widetilde{b}X_s + \beta)\,\mathrm{d}s + \int_0^t \sqrt{2cX_s^+}\,\mathrm{d}W_s$$
$$+ \int_0^t \int_0^\infty \int_0^{X_{s-}} z\,\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_0^\infty z\,M(\mathrm{d}s, \mathrm{d}z), \qquad t \in \mathbb{R}_+,$$

where

- $\widetilde{b} := b + \int_1^\infty (z-1)\,\mu(\mathrm{d}z)$,
- $(W_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process,
- N and M are Poisson random measures on $\mathbb{R}_{++}^3$ and $\mathbb{R}_{++}^2$ with intensity measures $\mathrm{d}s\,\mu(\mathrm{d}z)\,\mathrm{d}u$ and $\mathrm{d}s\,\nu(\mathrm{d}z)$,
- $\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) := N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) - \mathrm{d}s\,\mu(\mathrm{d}z)\,\mathrm{d}u$,
- $(W_t)_{t\in\mathbb{R}_+}$, N and M are independent,

and the solution is a CBI process with parameter vector (b, c, μ, β, ν).

Expectation of a CBI(b, c, μ, β, ν) process if $\int_1^\infty z\,\nu(\mathrm{d}z) < \infty$

$$\mathbb{E}(X_t \mid X_0 = x) = \mathrm{e}^{\widetilde{b}t} x + \widetilde{\beta} \int_0^t \mathrm{e}^{\widetilde{b}u}\,\mathrm{d}u, \qquad x \in \mathbb{R}_+, \quad t \in \mathbb{R}_+,$$

with $\widetilde{\beta} := \beta + \int_0^\infty z\,\nu(\mathrm{d}z)$.

Interpretation of $\mathrm{e}^{\widetilde{b}}$: branching mean

$$\mathrm{e}^{\widetilde{b}} = \mathbb{E}(Y_1 \mid Y_0 = 1),$$

where $(Y_t)_{t\in\mathbb{R}_+}$ is a CBI$(b, c, \mu, 0, 0)$ process, which can be considered as a pure branching process (without immigration).

$-\widetilde{b}$ can also be considered as the death rate

Interpretation of $\widetilde{\beta}$: immigration mean

$$\widetilde{\beta} = \mathbb{E}(Z_1 \mid Z_0 = 0),$$

where $(Z_t)_{t\in\mathbb{R}_+}$ is a CBI$(0, 0, 0, \beta, \nu)$ process, which can be considered as a pure immigration process (without branching).

Asymptotics of the expectation if $\int_1^\infty z\,\nu(\mathrm{d}z) < \infty$

- $\lim_{t\to\infty} \mathbb{E}(X_t \,|\, X_0 = x) = -\frac{\widetilde{\beta}}{\widetilde{b}}$ if $\widetilde{b} < 0$ (subcritical case);
- $\lim_{t\to\infty} t^{-1}\, \mathbb{E}(X_t \,|\, X_0 = x) = \widetilde{\beta}$ if $\widetilde{b} = 0$ (critical case);
- $\lim_{t\to\infty} \mathrm{e}^{-\widetilde{b}t}\, \mathbb{E}(X_t \,|\, X_0 = x) = x + \frac{\widetilde{\beta}}{\widetilde{b}}$ if $\widetilde{b} > 0$ (supercritical case).

Asymptotics of a subcritical or critical single-type CBI process (Li, 2011)

Let $(X_t)_{t\in\mathbb{R}_+}$ be a CBI(b, c, μ, β, ν) process such that $\mathbb{E}(X_0) < \infty$, $b \leqslant 0$, $\int_1^\infty z\,\nu(\mathrm{d}z) < \infty$ and $\widetilde{\beta} > 0$. Then $X_t \xrightarrow{\mathcal{D}} \pi$ as $t \to \infty$ with a probability distribution π if and only if

$$\exists x_0 \in \mathbb{R}_{++} \qquad \text{with} \qquad \int_0^{x_0} \frac{\psi(\lambda)}{\varphi(\lambda)}\,\mathrm{d}\lambda < \infty,$$

where

$$\varphi(\lambda) := c\lambda^2 - b\lambda + \int_0^\infty \left(\mathrm{e}^{-\lambda z} - 1 + \lambda(1 \wedge z)\right)\mu(\mathrm{d}z),$$

$$\psi(\lambda) := \beta\lambda + \int_0^\infty \left(1 - \mathrm{e}^{-\lambda r}\right)\nu(\mathrm{d}r).$$

If this holds, then the Laplace transform of π is given by

$$\int_0^\infty \mathrm{e}^{-x\lambda}\,\pi(\mathrm{d}\lambda) = \int_0^x \frac{\psi(\lambda)}{\varphi(\lambda)}\,\mathrm{d}\lambda, \qquad x \in \mathbb{R}_+.$$

Asymptotics of a critical single-type CBI process (Huang, Ma & Zhu, 2011; Barczy, Döring, Li & P, 2013)

Let $(X_t)_{t\in\mathbb{R}_+}$ be a $\mathrm{CBI}(b, c, \mu, \beta, \nu)$ process such that $\mathbb{E}(X_0) < \infty$, $b = 0$, $\int_1^\infty z^2 \, \mu(\mathrm{d}z) < \infty$ and $\int_1^\infty z \, \nu(\mathrm{d}z) < \infty$. Then

$$(\mathcal{X}_t^{(T)})_{t\in\mathbb{R}_+} := (T^{-1} X_{Tt})_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (\mathcal{X}_t)_{t\in\mathbb{R}_+} \qquad \text{as} \quad T \to \infty,$$

where $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is the pathwise unique strong solution of the SDE

$$\mathrm{d}\mathcal{X}_t = \widetilde{\beta} \, \mathrm{d}t + \sqrt{\widetilde{c} \mathcal{X}_t^+} \, \mathrm{d}\mathcal{W}_t, \quad t \in \mathbb{R}_+, \qquad \mathcal{X}_0 = 0,$$

with

$$\widetilde{c} := 2c + \int_0^\infty z^2 \, \mu(\mathrm{d}z) = \mathrm{Var}(Y_1 \mid Y_0 = 1),$$

where $(Y_t)_{t\in\mathbb{R}_+}$ is a $\mathrm{CBI}(0, c, \mu, 0, 0)$ (critical pure braching) process.

In fact, $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is a $\mathrm{CBI}(0, \widetilde{c}, 0, \widetilde{\beta}, 0)$ process, called Feller diffusion.

Asymptotics of a supercritical single-type CBI process (Li 2011; Kyprianou, Palau & Ren 2018; Barczy, Palau & P 2018)

Let $(X_t)_{t\in\mathbb{R}_+}$ be a CBI(b, c, μ, β, ν) process such that $\mathbb{E}(X_0) < \infty$, $b > 0$ and $\int_1^\infty z\,\nu(\mathrm{d}z) < \infty$.

(i) Then there is a non-negative random variable w_{X_0} with $\mathbb{E}(w_{X_0}) < \infty$ such that

$$\mathrm{e}^{-\widetilde{b}t} X_t \xrightarrow{\text{a.s.}} w_{X_0} \qquad \text{as } t \to \infty.$$

(ii) If, in addition, $\int_1^\infty z\log(z)\,\mu(\mathrm{d}z) < \infty$, then $\mathrm{e}^{-\widetilde{b}t} X_t \xrightarrow{L_1} w_{X_0}$ as $t \to \infty$, and $w_{X_0} \stackrel{\text{a.s.}}{=} 0$ if and only if $X_0 = 0$ and $\widetilde{\beta} = 0$ (equivalently, $X_t \stackrel{\text{a.s.}}{=} 0$ for all $t \in \mathbb{R}_+$).

(iii) If, in addition, $\int_1^\infty z\log(z)\,\mu(\mathrm{d}z) < \infty$ and $\widetilde{\beta} = 0$, then

$$\mathbb{P}(w_{X_0} = 0) = \mathbb{P}(\text{extinction time is finite}).$$

(iv) If, in addition, $\int_1^\infty z\log(z)\,\mu(\mathrm{d}z) = \infty$, then $w_{X_0} \stackrel{\text{a.s.}}{=} 0$.

Multi-type CBI process with parameter $(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$

Conservative time-homogeneous Markov process $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ with state space $\mathbb{R}_+^d$ and with infinitesimal generator

$$(\mathcal{A}f)(\boldsymbol{x}) = \langle \boldsymbol{\beta} + \boldsymbol{B}\boldsymbol{x}, \boldsymbol{f}'(\boldsymbol{x})\rangle + \sum_{i=1}^d c_i x_i f_{i,i}''(\boldsymbol{x}) + \int_{\mathcal{U}_d} \big[f(\boldsymbol{x}+\boldsymbol{z}) - f(\boldsymbol{x})\big]\,\nu(\mathrm{d}\boldsymbol{z})$$

$$+ \sum_{i=1}^d x_i \int_{\mathcal{U}_d} \big[f(\boldsymbol{x}+\boldsymbol{z}) - f(\boldsymbol{x}) - f_i'(\boldsymbol{x})(1\wedge z_i)\big]\,\mu_i(\mathrm{d}\boldsymbol{z})$$

for $f \in \mathbb{C}_c^2(\mathbb{R}_+^d, \mathbb{R})$ and $\boldsymbol{x} \in \mathbb{R}_+^d$, where $\boldsymbol{B} \in \mathbb{R}_{(+)}^{d\times d}$, $\boldsymbol{\beta}, \boldsymbol{c} \in \mathbb{R}_+^d$, ν is a Borel measure on $\mathcal{U}_d := \mathbb{R}_+^d \setminus \{\boldsymbol{0}\}$ satisfying $\int_{\mathcal{U}_d} (1 \wedge \|\boldsymbol{z}\|)\,\nu(\mathrm{d}\boldsymbol{z}) < \infty$, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d)$, where, for each $i \in \{1, \dots, d\}$, μ_i is a Borel measure on $\mathcal{U}_d$ satisfying

$$\int_{\mathcal{U}_d} \Bigg[(\|\boldsymbol{z}\| \wedge \|\boldsymbol{z}\|^2) + \sum_{j\in\{1,\dots,d\}\setminus\{i\}} (1\wedge z_j)\Bigg]\,\mu_i(\mathrm{d}\boldsymbol{z}) < \infty.$$

SDE of a MCBI process (Barczy, Li & P, 2015)

If $\int_{\mathcal{U}_d} \|\boldsymbol{z}\| \, \nu(\mathrm{d}z) < \infty$ then $\exists_1$ non-negative strong solution to the SDE

$$\boldsymbol{X}_t = \boldsymbol{X}_0 + \int_0^t (\widetilde{\boldsymbol{B}} \boldsymbol{X}_s + \boldsymbol{\beta}) \, \mathrm{d}s + \sum_{i=1}^d \boldsymbol{e}_i \int_0^t \sqrt{2c_i X_{s,i}^+} \, \mathrm{d}W_{s,i}$$

$$+ \sum_{j=1}^d \int_0^t \int_{\mathcal{U}_d} \int_0^{X_{s-,j}} \boldsymbol{z} \, \widetilde{N}_j(\mathrm{d}s, \mathrm{d}\boldsymbol{z}, \mathrm{d}u) + \int_0^t \int_{\mathcal{U}_d} \boldsymbol{z} \, M(\mathrm{d}s, \mathrm{d}\boldsymbol{z}), \qquad t \in \mathbb{R}_+,$$

where

- $\widetilde{\boldsymbol{B}} := (\widetilde{b}_{i,j})_{i,j \in \{1,\ldots,d\}} \in \mathbb{R}^{d \times d}_{(+)}$, $\widetilde{b}_{i,j} := b_{i,j} + \int_{\mathcal{U}_d} (z_i - \delta_{i,j})^+ \, \mu_j(\mathrm{d}\boldsymbol{z})$,
- $(\boldsymbol{W}_t)_{t \in \mathbb{R}_+}$ is a d-dimensional standard Wiener process,
- $N_1, \ldots, N_d$ and M are Poisson random measures on $\mathbb{R}_{++} \times \mathcal{U}_d \times \mathbb{R}_{++}$ and $\mathbb{R}_{++} \times \mathcal{U}_d$ with intensity measures $\mathrm{d}s \, \mu_j(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u$ and $\mathrm{d}s \, \nu(\mathrm{d}\boldsymbol{z})$,
- $\widetilde{N}_j(\mathrm{d}s, \mathrm{d}\boldsymbol{z}, \mathrm{d}u) := N_j(\mathrm{d}s, \mathrm{d}\boldsymbol{z}, \mathrm{d}u) - \mathrm{d}s \, \mu(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u$, $j \in \{1, \ldots, d\}$,
- $(\boldsymbol{W}_t)_{t \in \mathbb{R}_+}$, $N_1, \ldots, N_d$ and M are independent,

and the solution is a CBI process with parameter $(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$.

Expectation of an MCBI$(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$ process

$$\mathbb{E}(\boldsymbol{X}_t \,|\, \boldsymbol{X}_0 = \boldsymbol{x}) = \mathrm{e}^{t\widetilde{\boldsymbol{B}}}\boldsymbol{x} + \int_0^t \mathrm{e}^{u\widetilde{\boldsymbol{B}}}\widetilde{\boldsymbol{\beta}}\,\mathrm{d}u, \qquad \boldsymbol{x} \in \mathbb{R}_+^d, \quad t \in \mathbb{R}_+,$$

with $\widetilde{\boldsymbol{\beta}} := \boldsymbol{\beta} + \int_{\mathcal{U}_d} \boldsymbol{z}\,\nu(\mathrm{d}\boldsymbol{z})$.

Interpretation of $\mathrm{e}^{\widetilde{\boldsymbol{B}}}$: branching mean matrix

$$\mathrm{e}^{\widetilde{\boldsymbol{B}}}\boldsymbol{e}_j = \mathbb{E}(\boldsymbol{Y}_1 \,|\, \boldsymbol{Y}_0 = \boldsymbol{e}_j), \qquad j \in \{1, \ldots, d\},$$

where $(\boldsymbol{Y}_t)_{t\in\mathbb{R}_+}$ is an MCBI$(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{0}, 0)$ process, which can be considered as a pure branching process (without immigration).

Interpretation of $\boldsymbol{\beta}$: immigration mean vector

$$\widetilde{\boldsymbol{\beta}} = \mathbb{E}(\boldsymbol{Z}_1 \,|\, \boldsymbol{Z}_0 = \boldsymbol{0}),$$

where $(\boldsymbol{Z}_t)_{t\in\mathbb{R}_+}$ is an MCBI$(d, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{\beta}, \nu)$ process, which can be considered as a pure immigration process (without branching).

Irreducibility of a matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$

A matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$ is called reducible if there exist a permutation matrix $\boldsymbol{P} \in \mathbb{R}^{d\times d}$ and an integer r with $1 \leqslant r \leqslant d-1$ such that

$$\boldsymbol{P}^\top \boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{0} & \boldsymbol{A}_3 \end{bmatrix},$$

where $\boldsymbol{A}_1 \in \mathbb{R}^{r\times r}$, $\boldsymbol{A}_3 \in \mathbb{R}^{(d-r)\times(d-r)}$, $\boldsymbol{A}_2 \in \mathbb{R}^{r\times(d-r)}$, and $\boldsymbol{0} \in \mathbb{R}^{(d-r)\times r}$ is a null matrix. A matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$ is called irreducible if it is not reducible. (Hence 1-by-1 matrices are irreducible.)

$\mathrm{e}^{t\widetilde{\boldsymbol{B}}} \in \mathbb{R}_{+}^{d \times d}$ for all $t \in \mathbb{R}_{+}$.

The following statements are equivalent:

- $\exists\ t_0 \in \mathbb{R}_{++} := (0, \infty)$ with $\mathrm{e}^{t_0\widetilde{\boldsymbol{B}}} \in \mathbb{R}_{++}^{d \times d}$;
- $\forall\ t \in \mathbb{R}_{++}$ we have $\mathrm{e}^{t\widetilde{\boldsymbol{B}}} \in \mathbb{R}_{++}^{d \times d}$;
- $\mathrm{e}^{\widetilde{\boldsymbol{B}}}$ is irreducible;
- $\widetilde{\boldsymbol{B}}$ is irreducible.

Irreducibility

Let $(\boldsymbol{X}_t)_{t \in \mathbb{R}_+}$ be an MCBI$(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \beta, \nu)$. Then $(\boldsymbol{X}_t)_{t \in \mathbb{R}_+}$ is called irreducible if $\widetilde{\boldsymbol{B}}$ is irreducible.

For a matrix $\boldsymbol{A} \in \mathbb{R}^{d \times d}$, put

$$\sigma(\boldsymbol{A}) := \text{set of the eigenvalues of } \boldsymbol{A},$$

$$r(\boldsymbol{A}) := \max_{\lambda \in \sigma(\boldsymbol{A})} |\lambda| \qquad \text{(spectral radius of } \boldsymbol{A}),$$

$$s(\boldsymbol{A}) := \max_{\lambda \in \sigma(\boldsymbol{A})} \operatorname{Re}(\lambda) = \log r(\mathrm{e}^{\boldsymbol{A}}) \qquad \text{(by spectral mapping theorem).}$$

Asymptotics of the expectation

- $\lim_{t \to \infty} \mathbb{E}(\boldsymbol{X}_t \,|\, \boldsymbol{X}_0 = \boldsymbol{x}) = -\widetilde{\boldsymbol{B}}^{-1} \widetilde{\boldsymbol{\beta}}$ if $s(\widetilde{\boldsymbol{B}}) < 0$ (subcritical case);
- $\lim_{t \to \infty} t^{-1} \mathbb{E}(\boldsymbol{X}_t \,|\, \boldsymbol{X}_0 = \boldsymbol{x}) = \boldsymbol{\Pi} \widetilde{\boldsymbol{\beta}}$ if $s(\widetilde{\boldsymbol{B}}) = 0$ (critical case);
- $\lim_{t \to \infty} \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t} \mathbb{E}(\boldsymbol{X}_t \,|\, \boldsymbol{X}_0 = \boldsymbol{x}) = \boldsymbol{\Pi} \boldsymbol{x} + \frac{1}{s(\widetilde{\boldsymbol{B}})} \boldsymbol{\Pi} \widetilde{\boldsymbol{\beta}}$ if $s(\widetilde{\boldsymbol{B}}) > 0$ (supercritical case),

with $\boldsymbol{\Pi} := \widetilde{\boldsymbol{u}} \boldsymbol{u}^\top \in \mathbb{R}^{d \times d}_{++}$, where $\widetilde{\boldsymbol{u}}$ and $\boldsymbol{u}$ are the right and left Perron eigenvectors of $\widetilde{\boldsymbol{B}}$, corresponding to the eigenvalue $s(\widetilde{\boldsymbol{B}})$.

Asymptotics of a critical MCBI process (Barczy & P, 2014)

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be an irreducible and critical $\mathrm{MCBI}(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$ process such that $\mathbb{E}(\|\boldsymbol{X}_0\|^4) < \infty$, $\sum_{\ell=1}^d \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^4 \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \mu_\ell(\mathrm{d}\boldsymbol{z}) < \infty$ and $\int_{\mathcal{U}_d} \|\boldsymbol{r}\|^4 \mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant 1\}} \nu(\mathrm{d}\boldsymbol{r}) < \infty$. Then

$$(\boldsymbol{\mathcal{X}}_t^{(n)})_{t\in\mathbb{R}_+} := (n^{-1}\boldsymbol{X}_{\lfloor nt\rfloor})_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (\boldsymbol{\mathcal{X}}_t)_{t\in\mathbb{R}_+} := (\mathcal{X}_t \widetilde{\boldsymbol{u}})_{t\in\mathbb{R}_+}$$

as $n \to \infty$, where $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is the unique strong solution of the SDE

$$\mathrm{d}\mathcal{X}_t = \langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}}\rangle \,\mathrm{d}t + \sqrt{\langle \widetilde{\boldsymbol{C}}\boldsymbol{u}, \boldsymbol{u}\rangle \mathcal{X}_t^+}\,\mathrm{d}\mathcal{W}_t, \quad t \in \mathbb{R}_+, \qquad \mathcal{X}_0 = 0,$$

where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process and

$$\widetilde{\boldsymbol{C}} := \sum_{\ell=1}^d \langle \boldsymbol{e}_\ell, \widetilde{\boldsymbol{u}}\rangle \left(2c_\ell \boldsymbol{e}_\ell \boldsymbol{e}_\ell^\top + \int_{\mathcal{U}_d} \boldsymbol{z}\boldsymbol{z}^\top \mu_\ell(\mathrm{d}\boldsymbol{z})\right) = \mathrm{Var}(\boldsymbol{Y}_1 \mid \boldsymbol{Y}_0 = \widetilde{\boldsymbol{u}}),$$

where $(\boldsymbol{Y}_t)_{t\in\mathbb{R}_+}$ is an $\mathrm{MCBI}(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{0}, 0)$ (pure branching) process.

In fact, $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is a $\mathrm{CBI}(0, \langle \widetilde{\boldsymbol{C}}\boldsymbol{u}, \boldsymbol{u}\rangle, 0, \langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}}\rangle, 0)$ process, which is a Feller diffusion.

Asymptotics of a supercritical MCBI process (Kyprianou, Palau & Ren 2018; Barczy, Palau & P, 2018)

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be an irreducible and supercritical $\mathrm{MCBI}(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$ process such that $\mathbb{E}(\|\boldsymbol{X}_0\|) < \infty$ and $\int_{\mathcal{U}_d} \|\boldsymbol{r}\| \mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant 1\}} \, \nu(\mathrm{d}\boldsymbol{r}) < \infty$.

(i) Then there is a non-negative random variable $w_{\boldsymbol{u},\boldsymbol{X}_0}$ with $\mathbb{E}(w_{\boldsymbol{u},\boldsymbol{X}_0}) < \infty$ such that

$$\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t} \boldsymbol{X}_t \xrightarrow{\text{a.s.}} w_{\boldsymbol{u},\boldsymbol{X}_0} \widetilde{\boldsymbol{u}} \qquad \text{as } t \to \infty.$$

(ii) If, in addition, $\sum_{\ell=1}^d \int_{\mathcal{U}_d} \|\boldsymbol{z}\| \log(\|\boldsymbol{z}\|) \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \, \mu_\ell(\mathrm{d}\boldsymbol{z}) < \infty$, then $\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t} \boldsymbol{X}_t \xrightarrow{L_1} w_{\boldsymbol{u},\boldsymbol{X}_0}$ as $t \to \infty$, and $w_{\boldsymbol{u},\boldsymbol{X}_0} \stackrel{\text{a.s.}}{=} 0$ if and only if $\boldsymbol{X}_0 = \boldsymbol{0}$ and $\widetilde{\boldsymbol{\beta}} = \boldsymbol{0}$ (equivalently, $\boldsymbol{X}_t \stackrel{\text{a.s.}}{=} \boldsymbol{0}$ for all $t \in \mathbb{R}_+$).

(iii) If, in addition, $\sum_{\ell=1}^d \int_{\mathcal{U}_d} \|\boldsymbol{z}\| \log(\|\boldsymbol{z}\|) \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \, \mu_\ell(\mathrm{d}\boldsymbol{z}) = \infty$, then $w_{\boldsymbol{u},\boldsymbol{X}_0} \stackrel{\text{a.s.}}{=} \boldsymbol{0}$.

Asymptotics of projections of a supercritical MCBI process (Barczy, Palau & P, 2018)

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be an irreducible and supercritical MCBI$(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$ process such that $\mathbb{E}(\|\boldsymbol{X}_0\|) < \infty$ and $\int_{\mathcal{U}_d} \|\boldsymbol{r}\| \mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant 1\}} \nu(\mathrm{d}\boldsymbol{r}) < \infty$. Let $\lambda \in \sigma(\widetilde{\boldsymbol{B}})$ and let $\boldsymbol{v} \in \mathbb{C}^d$ be a left eigenvector of $\widetilde{\boldsymbol{B}}$ corresponding to the eigenvalue λ.

(i) If $\mathrm{Re}(\lambda) \in \left(\frac{1}{2}s(\widetilde{\boldsymbol{B}}), s(\widetilde{\boldsymbol{B}})\right]$ and the moment condition

$$\sum_{\ell=1}^{d} \int_{\mathcal{U}_d} g(\|\boldsymbol{z}\|) \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \mu_\ell(\mathrm{d}\boldsymbol{z}) < \infty$$

with

$$g(x) := \begin{cases} x^{\frac{s(\widetilde{\boldsymbol{B}})}{\mathrm{Re}(\lambda)}} & \text{if } \mathrm{Re}(\lambda) \in \left(\frac{1}{2}s(\widetilde{\boldsymbol{B}}), s(\widetilde{\boldsymbol{B}})\right), \\ x\log(x) & \text{if } \mathrm{Re}(\lambda) = s(\widetilde{\boldsymbol{B}}) \ (\Longleftrightarrow \lambda = s(\widetilde{\boldsymbol{B}})), \end{cases} \qquad x \in [1, \infty),$$

holds, then there exists a complex random variable $w_{\boldsymbol{v},\boldsymbol{X}_0}$ with $\mathbb{E}(|w_{\boldsymbol{v},\boldsymbol{X}_0}|) < \infty$ such that

$$\mathrm{e}^{-\lambda t}\langle \boldsymbol{v}, \boldsymbol{X}_t\rangle \to w_{\boldsymbol{v},\boldsymbol{X}_0} \qquad \text{as } t \to \infty \text{ in } L_1 \text{ and almost surely.}$$

(ii) If $\mathrm{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}})$ and the moment condition

$$\int_{\mathcal{U}_d} \|\boldsymbol{r}\|^2 \mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant 1\}} \, \nu(\mathrm{d}\boldsymbol{z}) + \sum_{\ell=1}^{d} \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^4 \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \, \mu_\ell(\mathrm{d}\boldsymbol{z}) < \infty \qquad (1)$$

holds, then

$$t^{-1/2} \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t/2} \begin{pmatrix} \mathrm{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \mathrm{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \xrightarrow{\mathcal{D}} \sqrt{w_{\boldsymbol{u},\boldsymbol{X}_0}} \, \boldsymbol{Z}_{\boldsymbol{v}} \qquad \text{as } t \to \infty,$$

where $\boldsymbol{Z}_{\boldsymbol{v}}$ is a 2-dimensional random vector with $\boldsymbol{Z}_{\boldsymbol{v}} \stackrel{\mathcal{D}}{=} \mathcal{N}_2(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{v}})$ independent of $w_{\boldsymbol{u},\boldsymbol{X}_0}$, where

$$\boldsymbol{\Sigma}_{\boldsymbol{v}} := \frac{1}{2} \sum_{\ell=1}^{d} \langle \boldsymbol{e}_\ell, \widetilde{\boldsymbol{u}} \rangle \left(C_{\boldsymbol{v},\ell} \boldsymbol{I}_2 + \begin{pmatrix} \mathrm{Re}(\widetilde{C}_{\boldsymbol{v},\ell}) & \mathrm{Im}(\widetilde{C}_{\boldsymbol{v},\ell}) \\ \mathrm{Im}(\widetilde{C}_{\boldsymbol{v},\ell}) & -\mathrm{Re}(\widetilde{C}_{\boldsymbol{v},\ell}) \end{pmatrix} \mathbb{1}_{\{\mathrm{Im}(\lambda)=0\}} \right)$$

with

$$C_{\boldsymbol{v},\ell} := 2|\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle|^2 c_\ell + \int_{\mathcal{U}_d} |\langle \boldsymbol{v}, \boldsymbol{z} \rangle|^2 \, \mu_\ell(\mathrm{d}\boldsymbol{z}), \qquad \ell \in \{1, \ldots, d\},$$

$$\widetilde{C}_{\boldsymbol{v},\ell} := 2\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle^2 c_\ell + \int_{\mathcal{U}_d} \langle \boldsymbol{v}, \boldsymbol{z} \rangle^2 \, \mu_\ell(\mathrm{d}\boldsymbol{z}), \qquad \ell \in \{1, \ldots, d\}.$$

(iii) If $\operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\right)$ and the moment condition (1) holds, then

$$e^{-s(\widetilde{\boldsymbol{B}})t/2} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \xrightarrow{\mathcal{D}} \sqrt{w_{\boldsymbol{u},\boldsymbol{X}_0}}\, \boldsymbol{Z}_{\boldsymbol{v}} \qquad \text{as} \quad t \to \infty,$$

where $\boldsymbol{Z}_{\boldsymbol{v}}$ is a 2-dimensional random vector with $\boldsymbol{Z}_{\boldsymbol{v}} \stackrel{\mathcal{D}}{=} \mathcal{N}_2(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{v}})$ independent of $w_{\boldsymbol{u},\boldsymbol{X}_0}$, where

$$\begin{aligned} \boldsymbol{\Sigma}_{\boldsymbol{v}} := \frac{1}{2} \sum_{\ell=1}^{d} \frac{\langle \boldsymbol{e}_\ell, \widetilde{\boldsymbol{u}} \rangle C_{\boldsymbol{v},\ell}}{s(\widetilde{\boldsymbol{B}}) - 2\operatorname{Re}(\lambda)} \boldsymbol{I}_2 \\ + \frac{1}{2} \sum_{\ell=1}^{d} \langle \boldsymbol{e}_\ell, \widetilde{\boldsymbol{u}} \rangle \begin{pmatrix} \operatorname{Re}\left(\frac{\widetilde{C}_{\boldsymbol{v},\ell}}{s(\widetilde{\boldsymbol{B}})-2\lambda}\right) & \operatorname{Im}\left(\frac{\widetilde{C}_{\boldsymbol{v},\ell}}{s(\widetilde{\boldsymbol{B}})-2\lambda}\right) \\ \operatorname{Im}\left(\frac{\widetilde{C}_{\boldsymbol{v},\ell}}{s(\widetilde{\boldsymbol{B}})-2\lambda}\right) & -\operatorname{Re}\left(\frac{\widetilde{C}_{\boldsymbol{v},\ell}}{s(\widetilde{\boldsymbol{B}})-2\lambda}\right) \end{pmatrix}. \end{aligned}$$

Asymptotics of projections of a supercritical MCBI process with random scalings (Barczy, Palau & P, 2018)

Suppose that the assumptions of the earlier Theorem hold and $\widetilde{\beta} \neq \mathbf{0}$.

(i) If $\operatorname{Re}(\lambda) \in \left(\frac{1}{2}s(\widetilde{\boldsymbol{B}}), s(\widetilde{\boldsymbol{B}})\right]$, then, as $t \to \infty$,

$$\frac{\mathbb{1}_{\{\boldsymbol{X}_t \neq \mathbf{0}\}}}{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle^{\operatorname{Re}(\lambda)/s(\widetilde{\boldsymbol{B}})}} \begin{pmatrix} \cos(\operatorname{Im}(\lambda)t) & \sin(\operatorname{Im}(\lambda)t) \\ -\sin(\operatorname{Im}(\lambda)t) & \cos(\operatorname{Im}(\lambda)t) \end{pmatrix} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix}$$

$$\xrightarrow{\text{a.s.}} \frac{1}{w_{\boldsymbol{u},\boldsymbol{X}_0}^{\operatorname{Re}(\lambda)/s(\widetilde{\boldsymbol{B}})}} \begin{pmatrix} \operatorname{Re}(w_{\boldsymbol{v},\boldsymbol{X}_0}) \\ \operatorname{Im}(w_{\boldsymbol{v},\boldsymbol{X}_0}) \end{pmatrix}.$$

(ii) If $\operatorname{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}})$, then, as $t \to \infty$,

$$\frac{\mathbb{1}_{\{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle > 1\}}}{\sqrt{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle \log(\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle)}} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2\left(\mathbf{0}, \frac{1}{s(\widetilde{\boldsymbol{B}})} \boldsymbol{\Sigma}_{\boldsymbol{v}}\right).$$

(iii) If $\operatorname{Re}(\lambda) \in \left(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\right)$, then, as $t \to \infty$,

$$\frac{\mathbb{1}_{\{\boldsymbol{X}_t \neq \mathbf{0}\}}}{\sqrt{\langle \boldsymbol{u}, \boldsymbol{X}_t \rangle}} \begin{pmatrix} \operatorname{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \\ \operatorname{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle) \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{v}}).$$

Relative frequencies of distinct types of individuals

Critical case (Barczy & P, 2016)

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a critical and irreducible MCBI$(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$ process such that $\mathbb{E}(\|\boldsymbol{X}_0\|^4) < \infty$, $\int_{\mathcal{U}_d} \|\boldsymbol{r}\|^4 \mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant 1\}} \,\nu(\mathrm{d}\boldsymbol{r}) + \sum_{\ell=1}^d \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^4 \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \,\mu_\ell(\mathrm{d}\boldsymbol{r}) < \infty$ and $\widetilde{\boldsymbol{\beta}} \neq \boldsymbol{0}$. Then for each $i, j \in \{1, \ldots, d\}$, as $n \to \infty$, we have

$$\mathbb{1}_{\{\langle \boldsymbol{e}_j, \boldsymbol{X}_{\lfloor nt \rfloor}\rangle \neq 0\}} \frac{\langle \boldsymbol{e}_i, \boldsymbol{X}_{\lfloor nt \rfloor}\rangle}{\langle \boldsymbol{e}_j, \boldsymbol{X}_{\lfloor nt \rfloor}\rangle} \xrightarrow{\mathbb{P}} \frac{\langle \boldsymbol{e}_i, \widetilde{\boldsymbol{u}}\rangle}{\langle \boldsymbol{e}_j, \widetilde{\boldsymbol{u}}\rangle}, \qquad \mathbb{1}_{\{\boldsymbol{X}_{\lfloor nt \rfloor} \neq \boldsymbol{0}\}} \frac{\langle \boldsymbol{e}_i, \boldsymbol{X}_{\lfloor nt \rfloor}\rangle}{\sum\limits_{k=1}^d \langle \boldsymbol{e}_k, \boldsymbol{X}_{\lfloor nt \rfloor}\rangle} \xrightarrow{\text{a.s.}} \langle \boldsymbol{e}_i, \widetilde{\boldsymbol{u}}\rangle$$

Supercritical case (Barczy, Palau & P, 2018)

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a supercritical and irreducible MCBI$(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$ process such that $\mathbb{E}(\|\boldsymbol{X}_0\|) < \infty$, $\int_{\mathcal{U}_d} \|\boldsymbol{r}\| \mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant 1\}} \,\nu(\mathrm{d}\boldsymbol{r}) < \infty$ and $\widetilde{\boldsymbol{\beta}} \neq \boldsymbol{0}$. Then for each $i, j \in \{1, \ldots, d\}$, as $t \to \infty$, we have

$$\mathbb{1}_{\{\langle \boldsymbol{e}_j, \boldsymbol{X}_t\rangle \neq 0\}} \frac{\langle \boldsymbol{e}_i, \boldsymbol{X}_t\rangle}{\langle \boldsymbol{e}_j, \boldsymbol{X}_t\rangle} \xrightarrow{\text{a.s.}} \frac{\langle \boldsymbol{e}_i, \widetilde{\boldsymbol{u}}\rangle}{\langle \boldsymbol{e}_j, \widetilde{\boldsymbol{u}}\rangle}, \qquad \mathbb{1}_{\{\boldsymbol{X}_t \neq \boldsymbol{0}\}} \frac{\langle \boldsymbol{e}_i, \boldsymbol{X}_t\rangle}{\sum\limits_{k=1}^d \langle \boldsymbol{e}_k, \boldsymbol{X}_t\rangle} \xrightarrow{\text{a.s.}} \langle \boldsymbol{e}_i, \widetilde{\boldsymbol{u}}\rangle.$$

On the limit random variable $w_{\boldsymbol{v},\boldsymbol{X}_0}$ (Barczy, Palau & P, 2018)

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a supercritical and irreducible MCBI$(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$ process such that $\mathbb{E}(\|\boldsymbol{X}_0\|) < \infty$ and $\int_{\mathcal{U}_d} \|\boldsymbol{r}\| \mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant 1\}} \, \nu(\mathrm{d}\boldsymbol{r}) < \infty$. Let $\lambda \in \sigma(\widetilde{\boldsymbol{B}})$ be such that $\mathrm{Re}(\lambda) \in \big(\frac{1}{2} s(\widetilde{\boldsymbol{B}}), s(\widetilde{\boldsymbol{B}})\big]$ and $\sum_{\ell=1}^{d} \int_{\mathcal{U}_d} g(\|\boldsymbol{z}\|) \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \, \mu_\ell(\mathrm{d}\boldsymbol{z}) < \infty$, and let $\boldsymbol{v} \in \mathbb{C}^d$ be a left eigenvector of $\widetilde{\boldsymbol{B}}$ corresponding to the eigenvalue λ.

(i) If

 (a) $\widetilde{\boldsymbol{\beta}} \neq \boldsymbol{0}$, i.e., $\boldsymbol{\beta} \neq \boldsymbol{0}$ or $\nu \neq 0$,

 (b) $\nu(\{\boldsymbol{r} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{r}\rangle \neq 0\}) > 0$, or there exists $\ell \in \{1, \ldots, d\}$ such that $\langle \boldsymbol{v}, \boldsymbol{e}_\ell\rangle c_\ell \neq 0$ or $\mu_\ell(\{\boldsymbol{z} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{z}\rangle \neq 0\}) > 0$,

 then the law of $w_{\boldsymbol{v},\boldsymbol{X}_0}$ does not have atoms, thus $\mathbb{P}(w_{\boldsymbol{v},\boldsymbol{X}_0} = 0) = 0$.

(ii) If (b) does not hold, then $\mathbb{P}(w_{\boldsymbol{v},\boldsymbol{X}_0} = \langle \boldsymbol{v}, \boldsymbol{X}_0 + \lambda^{-1}\widetilde{\boldsymbol{\beta}}\rangle) = 1$.

(iii) If $\lambda = s(\widetilde{\boldsymbol{B}})$, $\boldsymbol{v} = \boldsymbol{u}$ and (a) holds, then $\mathbb{P}(w_{\boldsymbol{u},\boldsymbol{X}_0} = 0) = 0$.

(iv) If $\lambda = s(\widetilde{\boldsymbol{B}})$, $\boldsymbol{v} = \boldsymbol{u}$ and the conditions (a) and (b) do not hold, then $\mathbb{P}(w_{\boldsymbol{u},\boldsymbol{X}_0} = 0) = \mathbb{P}(\boldsymbol{X}_0 = 0)$.

Stochastic fixed point equation (Buraczewski, Damek & Mikosch)

Let $(\boldsymbol{A}, \boldsymbol{B})$ be a random element in $\mathbb{R}^{d\times d} \times \mathbb{R}^d$. Assume that

(i) $\boldsymbol{A}$ is invertible almost surely,

(ii) $\mathbb{P}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B} = \boldsymbol{x}) < 1$ for every $\boldsymbol{x} \in \mathbb{R}^d$,

(iiii) the d-dimensional fixed point equation $\boldsymbol{X} \stackrel{\mathcal{D}}{=} \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}$, where $(\boldsymbol{A}, \boldsymbol{B})$ and $\boldsymbol{X}$ are independent, has a solution $\boldsymbol{X}$, which is unique in distribution.

Then the distribution of $\boldsymbol{X}$ does not have atoms and is of pure type, i.e., it is either absolutely continuous or singular with respect to the Lebesgue measure in $\mathbb{R}^d$.

Corollary

Let $\boldsymbol{A} \in \mathbb{R}^{d\times d}$ with $\det(\boldsymbol{A}) \neq 0$ and $r(\boldsymbol{A}) < 1$. Let $\boldsymbol{B}$ be a d-dimensional non-deterministic random vector with $\mathbb{E}(\|\boldsymbol{B}\|) < \infty$.

Then the d-dimensional fixed point equation $\boldsymbol{X} \stackrel{\mathcal{D}}{=} \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}$, where $\boldsymbol{X}$ is independent of $\boldsymbol{B}$, has a solution $\boldsymbol{X}$ which is unique in distribution, the distribution of $\boldsymbol{X}$ does not have atoms and is of pure type.

Deterministic projections of MCBI processes (Barczy, Palau & P, 2018)

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be an MCBI$(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$ process such that $\mathbb{E}(\|\boldsymbol{X}_0\|) < \infty$ and $\int_{\mathcal{U}_d} \|\boldsymbol{r}\| \mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant 1\}} \nu(\mathrm{d}\boldsymbol{r}) < \infty$. Let $\lambda \in \sigma(\widetilde{\boldsymbol{B}})$, and let $\boldsymbol{v} \in \mathbb{C}^d$ be a left eigenvector of $\widetilde{\boldsymbol{B}}$ corresponding to the eigenvalue λ. Then the following three assertions are equivalent:

(i) There exists $t \in \mathbb{R}_{++}$ such that $\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ is deterministic.

(ii) One of the following two conditions holds:

  (a) $\mathbb{P}(\boldsymbol{X}_t = \boldsymbol{0}) = 1$ for all $t \in \mathbb{R}_+$.

  (b) $\langle \boldsymbol{v}, \boldsymbol{X}_0 \rangle$ is deterministic, $\langle \boldsymbol{v}, \boldsymbol{e}_\ell \rangle c_\ell = 0$ and $\mu_\ell(\{\boldsymbol{z} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \neq 0\}) = 0$ for every $\ell \in \{1, \ldots, d\}$, and $\nu(\{\boldsymbol{r} \in \mathcal{U}_d : \langle \boldsymbol{v}, \boldsymbol{r} \rangle \neq 0\}) = 0$.

(iii) For each $t \in \mathbb{R}_+$, $\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle$ is deterministic.

If $(\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle)_{t\in\mathbb{R}_+}$ is deterministic, then

$$\langle \boldsymbol{v}, \boldsymbol{X}_t \rangle \stackrel{\text{a.s.}}{=} \mathrm{e}^{\lambda t} \langle \boldsymbol{v}, \mathbb{E}(\boldsymbol{X}_0) \rangle + \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle \int_0^t \mathrm{e}^{\lambda u} \, \mathrm{d}u, \qquad t \in \mathbb{R}_+.$$

Variance matrix of the real and imaginary parts of the projection of an MCBI process (Barczy, Palau & P, 2018)

If $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ is a supercritical and irreducible MCBI$(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$ process such that $\mathbb{E}(\|\boldsymbol{X}_0\|^2) < \infty$ and $\int_{\mathcal{U}_d} \|\boldsymbol{r}\|^2 \mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant 1\}}\, \nu(\mathrm{d}\boldsymbol{r}) + \sum_{\ell=1}^{d} \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^2 \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}}\, \mu_\ell(\mathrm{d}\boldsymbol{r}) < \infty$, then for each left eigenvector $\boldsymbol{v} \in \mathbb{C}^d$ of $\widetilde{\boldsymbol{B}}$ corresponding to an arbitrary eigenvalue $\lambda \in \sigma(\widetilde{\boldsymbol{B}})$ with $\mathrm{Re}(\lambda) \in \big(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\big]$ we have

$$h(t)\,\mathbb{E}\left(\begin{pmatrix}\mathrm{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t\rangle)\\ \mathrm{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t\rangle)\end{pmatrix}\begin{pmatrix}\mathrm{Re}(\langle \boldsymbol{v}, \boldsymbol{X}_t\rangle)\\ \mathrm{Im}(\langle \boldsymbol{v}, \boldsymbol{X}_t\rangle)\end{pmatrix}^\top\right) \to \left(\langle \boldsymbol{u}, \mathbb{E}(\boldsymbol{X}_0)\rangle + \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}}\rangle}{s(\widetilde{\boldsymbol{B}})}\right)\boldsymbol{\Sigma}_{\boldsymbol{v}}$$

as $t \to \infty$, where the scaling factor $h : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is given by

$$h(t) := \begin{cases} \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t} & \text{if } \mathrm{Re}(\lambda) \in \big(-\infty, \frac{1}{2}s(\widetilde{\boldsymbol{B}})\big),\\ t^{-1}\mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t} & \text{if } \mathrm{Re}(\lambda) = \frac{1}{2}s(\widetilde{\boldsymbol{B}}),\\ \mathrm{e}^{-2\,\mathrm{Re}(\lambda)t} & \text{if } \mathrm{Re}(\lambda) \in \big(\frac{1}{2}s(\widetilde{\boldsymbol{B}}), s(\widetilde{\boldsymbol{B}})\big]. \end{cases}$$

A stable limit theorem for martingales (Küchler & Sørensen, 1997; Crimaldi & Pratelli, 2005)

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $(\boldsymbol{M}_t)_{t\in\mathbb{R}_+}$ be a d-dimensional martingale with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ such that it has càdlàg sample paths almost surely. Suppose that there exists a function $\boldsymbol{Q} : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ such that $\lim_{t\to\infty} \boldsymbol{Q}(t) = \boldsymbol{0}$,

$$\boldsymbol{Q}(t)[\boldsymbol{M}]_t \boldsymbol{Q}(t)^\top \xrightarrow{\mathbb{P}} \boldsymbol{\eta} \qquad \text{as} \quad t \to \infty,$$

where $\boldsymbol{\eta}$ is a $d \times d$ random (necessarily positive semidefinite) matrix and $([\boldsymbol{M}]_t)_{t\in\mathbb{R}_+}$ denotes the (matrix-valued) quadratic variation process of $(\boldsymbol{M}_t)_{t\in\mathbb{R}_+}$, and

$$\mathbb{E}\left(\sup_{u\in[0,t]} \|\boldsymbol{Q}(t)(\boldsymbol{M}_u - \boldsymbol{M}_{u-})\| \right) \to 0 \qquad \text{as} \quad t \to \infty.$$

Then, for each $\mathbb{R}^{k\times\ell}$-valued random matrix $\boldsymbol{A}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$,

$$(\boldsymbol{Q}(t)\boldsymbol{M}_t, \boldsymbol{A}) \xrightarrow{\mathcal{D}} (\boldsymbol{\eta}^{1/2}\boldsymbol{Z}, \boldsymbol{A}) \qquad \text{as} \quad t \to \infty,$$

where $\boldsymbol{Z}$ is a d-dimensional random vector with $\boldsymbol{Z} \stackrel{\mathcal{D}}{=} \mathcal{N}_d(\boldsymbol{0}, \boldsymbol{I}_d)$ independent of $(\boldsymbol{\eta}, \boldsymbol{A})$.

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