Iterated and sequential St.Petersburg games

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$$\mathbf{E}\{X\} = \sum_{k=1}^{\infty} 2^k 2^{-k} = \infty$$

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$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n \log_2 n} = 1$$

in probability,
where $\log_2$ denotes the logarithm with base 2, (Feller (1945)).

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Merging theorem:

$$\sup_{x \in \mathbb{R}} \left| \mathbf{P}\left\{ \frac{S_n}{n} - \log_2 n \le x \right\} - G_{\gamma_n}(x) \right| \to 0, \quad \text{as } n \to \infty,$$

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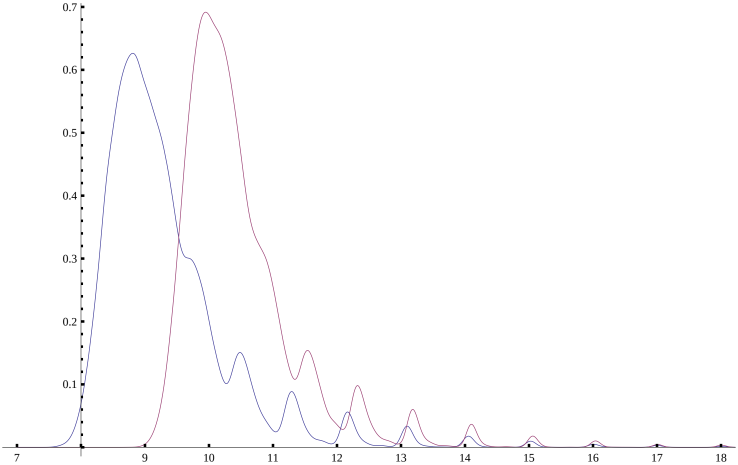
Parametric class of distributions $\{G_\gamma\}$.

The histograms of $\log_2 S_n$ for $n = 2^6$ and for $n = 2^7$.



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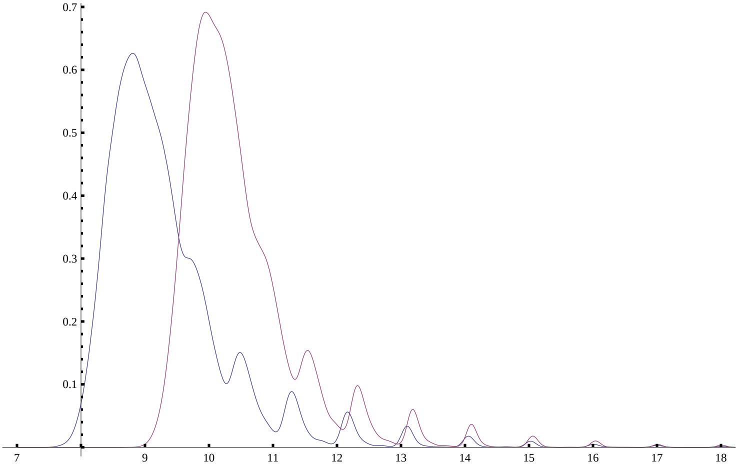
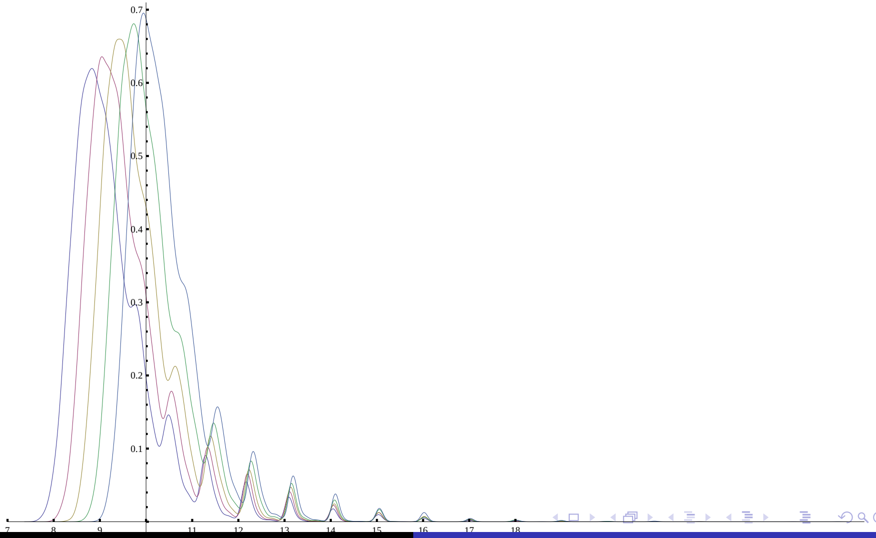


Surprise: $\mathbf{Var}(\log_2 S_n) = O(1/\ln n) \to 0$

The histograms of $\log_2 S_n$ for $n = 2^{6+\eta}$, $\eta = 0, 0.25, 0.5, 0.75, 1.$



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Merging theorem for the maximum:

$$\sup_{j \in \mathbb{Z}} \left| \mathbf{P}\left\{X_n^* = 2^{\lceil \log_2 n \rceil + j}\right\} - p_{j,\gamma_n} \right| = O(n^{-1}), \quad \text{as } n \to \infty$$

(Berkes, Csáki and Csörgő (1999)).

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j	-2	-1	0	1	2	3	4	5
$p_{j,1}$	0.018	0.117	0.233	0.239	0.172	0.104	0.057	0.03

Table: Limit distribution of $X_n^* = 2^{\lceil \log_2 n \rceil + j}$ with $\gamma = 1$.

Decomposition

Peter Kevei

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$$\mathbf{P}\left\{\frac{S_n}{n} - \log_2 n \le x\right\}$$

$$= \sum_{j=1-\lceil \log_2 n\rceil}^{\infty} \mathbf{P}\left\{\frac{S_n}{n} - \log_2 n \le x \middle| X_n^* = 2^{\lceil \log_2 n\rceil + j}\right\} \mathbf{P}\left\{X_n^* = 2^{\lceil \log_2 n\rceil + j}\right\}$$

$$\approx \sum_{j=-\infty}^{\infty} G_{j,\gamma_n}(x) p_{j,\gamma_n}$$

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$G_{j,\gamma}(x)$ has a density.

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small maximum:

$$X_n^* = 2^{\lceil \log_2 n \rceil + k_n}$$

$k_n \to -\infty$

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conditional merging theorem

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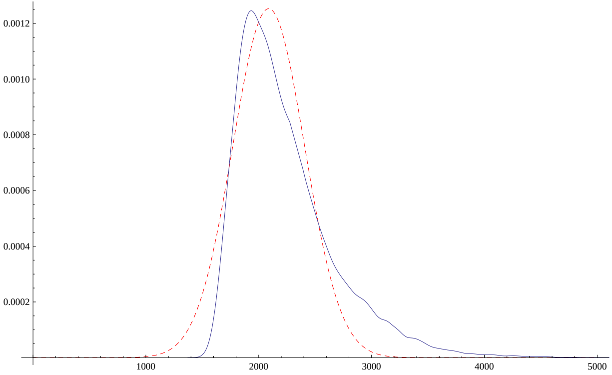
The histogram of S_n for $n = 2^7$ conditioned on $X_n^* = 2^{10}$ and a fitted Gaussian density.



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given $X_n^* = 2^{\lceil \log_2 n \rceil + k_n}$ with $k_n \to \infty$ we have

$$\frac{S_n}{X_n^*} \longrightarrow 1$$

in probability.

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In each step the player reinvest his capital with proportional cost.

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It means that after the n-th round the capital is

$$S_n^{(c)} = S_{n-1}^{(c)} X_n/4 = S_0 \prod_{i=1}^{n} (X_i/4) = \prod_{i=1}^{n} (X_i/4).$$

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$S_n^{(c)}$ has exponential trend:

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By induction, for n-th step of the portfolio game the initial capital is S_{n-1}, therefore

$$S_n = S_{n-1}(\mathbf{X}_n, \mathbf{b}) = \prod_{i=1}^{n} (\mathbf{X}_i, \mathbf{b}).$$

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$$\mathbf{b}^* = (0.385, 0.615)$$

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&\to \mathbf{E}\{\log_2(\mathbf{X}_1, \mathbf{b})\}
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$$\mathbf{b}^* = (0.385, 0.615)$$

and

$$W_1^* = W(0.385) = 0.149.$$

2 St. Petersburg components

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$$\mathbf{b}^* = (1/d, \ldots, 1/d, 0)$$

and the asymptotic average growth rate is

$$\begin{aligned} W_d^* &= \mathbf{E}\left\{\log_2\left(\frac{1}{d}\sum_{i=1}^{d}(X_i/4)\right)\right\} \\ &\approx \frac{\log_2 \log_2 d}{\ln 2 \, \log_2 d} + \log_2 \log_2 d - 2 \end{aligned}$$

For any (large) $c < 1$, there is a d such that

$$W_d^* \approx \log_2 \log_2 d + \log_2(1-c) > 0$$