Mushroom and Isola Bifurcations in a Within-Host Bacterial Infection Model with Allee Effect

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The interplay between nonlinear bacterial growth and host cell regulation can generate complex dynamics in within-host infection models. This work presents a model for a self-replicating bacterial infection incorporating an Allee effect, under both constant host cell turnover and density-dependent (logistic) proliferation. The analysis establishes well-posedness and identifies the basic reproduction number as a threshold governing the transition between bacteria-free and infection states. A detailed study of the equilibrium equation reveals how the Allee threshold ε shapes the number and feasibility of steady states, leading to multiple infection equilibria. In the logistic formulation, this interplay gives rise to intricate bifurcation structures, including mushroom and isola configurations, both of which exhibit bistability, and a cusp catastrophe, where two branches of the fold curve meet tangentially, highlighting the strong nonlinear effects induced by the Allee mechanism. The system undergoes a transcritical bifurcation at $\mathcal{R}_0 = 1$, with explicitly determined direction, while the logistic model further exhibits oscillatory dynamics through a Hopf bifurcation, as supported by numerical simulations. These results demonstrate how Allee effects and host cell regulation jointly enhance the dynamical complexity of within-host bacterial systems.