Softening of Locally Polyhedral Tilings

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A *tiling* of $\mathbb{R}^d$ is a covering of $\mathbb{R}^d$ with cells that do not overlap (i.e., which are disjoint in their interior), and whose union is the whole space $\mathbb{R}^d$. The most extensively studied tilings tessellate $\mathbb{R}^3$ by convex polyhedra; these form the *cells* of the tiling. In a *polyhedral tiling* of $\mathbb{R}^3$, we require that the cells be convex polyhedra, any two of which intersect only in common vertices, edges, or faces. Polyhedral tilings admit various generalizations. Following the work of Domokos, Goriely, G. Horváth, and Regős [2], our focus is on *polyhedric tilings*: here, $\mathbb{R}^d$ for $d = 3$ is tiled with cells that are compact topological d-balls whose pairwise intersections are lower-dimensional topological balls, and which satisfy additional combinatorial and smoothness conditions. Vertices, edges, and faces of the tiling are defined according to their topological dimension. The *nodes* of the tiling play a central role: these are points at which the number of containing cells is strictly locally maximal. Given a topological ball C and a point $p \in C$ on its boundary, p is called a *spike* of C if there is no C^2 smooth curve on ∂C that goes through p. The cell C is called *soft* if it does not have spikes. Finally, a tiling is *completely soft* if it is composed of soft cells. These notions apply in both two- and three-dimensional settings.

In the context of soft cells, the first natural question is whether completely soft tilings of $\mathbb{R}^d$ exist for $d = 2$ and $d = 3$. In 2023, Gábor Domokos, Ákos G. Horváth, and Krisztina Regős [1] resolved the planar analogue by proving that, in any tiling of the plane by polygons in which any two polygons intersect only in common vertices or edges, the cells must have at least 2 spikes on average, provided that the average exists. The three-dimensional setting is markedly different: there exist many polyhedric tilings consisting entirely of soft cells. Our focus is on whether a polyhedral tiling can be modified in such a way that all cells become smooth. Our main result verifies that this can indeed be achieved: answering and generalizing a conjecture formulated in [2], we prove that every locally polyhedral tiling of the space can be completely softened.

This is a joint work with Gergely Ambrus from University of Szeged and Alfréd Rényi Institute of Mathematics.

[1] G. Domokos, Á. G. Horváth, K. Regős, A two-vertex theorem for normal tilings, *Aequationes Mathematicae* **97(1)** (2023), 185–197.

[2] G. Domokos, A. Goriely, Á. G. Horváth, K. Regős, Soft cells and the geometry of seashells, *PNAS Nexus* **3(9)** (2024), 311.