Definable Preoders of Directed Graphs with Compatible Operations

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A preorder is a reflexive and transitive relation. On any reflexive directed graph $\mathbb{G}$ one may define various preorders, such as

(1) the transitive closure of $\mathbb{G}$,

(2) the transitive closure of $\mathbb{G} \cup \mathbb{G}^{-1}$, and

(3) the transitive closure of $\mathbb{G} \cap \mathbb{G}^{-1}$.

If we look at the components of these preorders, then these are precisely the strongly, weakly and extremely connected components of the digraph $\mathbb{G}$. The definitions we have used in these examples can be expressed by primitive positive formulas (e.g. only conjunctions and existential quantifiers are allowed). An alternative definition can be given using graph homomorphisms from a template digraph $\mathbb{H}$ with two distinguished vertices: s and t to the target graph $\mathbb{G}$, where the defined preorder is the transitive closure of $(\varphi(s), \varphi(t))$ for all graph homomorphisms $\varphi : \mathbb{H} \to \mathbb{G}$. This generalizes the various notions of connected components of a directed graph. We have studied these definable quasiorders (using the st-digraph $\mathbb{H}$) and established an order between all these notions. For example, the strong components of a $\mathbb{G}$ is always a subrelation of the weak components of $\mathbb{G}$. Similarly, given two st-digraphs $\mathbb{H}_1$ and $\mathbb{H}_2$ they may always define the same quasiorder, or one may always be a subrelation of the other, or they might not be related in general.

Our first result is the complete description of the order relation between the preorders defined by an st-digraph $\mathbb{H}$ with at most 3 vertices. We show that it is enough to core relational structures. We also show that this preorder on the set of st-graphs has a compatible semilattice operation but it is not lattice ordered. Two different st-graphs may define different preorders in general, but they may produce the same preorder on special classes of digraphs $\mathbb{G}$. We have studied this phenomenon for three classes of reflexive digraphs:

(1) digraphs having a compatible Maltsev-operation,

(2) digraphs having a compatible near-unanimity operation, and

(3) digraphs having a compatible semilattice operation.

For example every reflexive digraph with a compatible Maltsev operation is symmetric, thus the weak and strong components are always the same. It turns out that for Maltsev- and near-unanimity digraphs there are only finitely many definable classes of preorders. We show that for reflexive digraphs with semilattice operation there are infinitely many classes.