Qualitative Properties of Numerical Solutions for Nonlinear Elliptic and Parabolic Partial Differential Equations

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Discrete maximum principles and non-negativity preservation play a vital role in assessing the qualitative reliability of numerical methods for nonlinear partial differential equations [3, 4, 5, 6]. Motivated by their importance in modeling nonlinear diffusion and reaction-diffusion processes, this work presents a unified study of qualitative properties for both nonlinear elliptic and nonlinear parabolic problems. For the elliptic case, we consider a broad class of nonlinear boundary value problems that includes numerous relevant applications. We derive exactly computable geometric conditions on widely used finite element shapes–triangles, tetrahedra, rectangles, and prisms–that guarantee the validity of discrete maximum principles [1]. These results provide practical criteria for ensuring that numerical approximations respect fundamental qualitative features. Extending these ideas to time-dependent settings, we consider nonlinear parabolic reaction-diffusion equations and establish explicit sufficient conditions for the preservation of discrete non-negativity under finite element spatial discretization. Our analysis again covers common element geometries such as triangles, tetrahedra, rectangles, and prisms. In addition, we examine several time-stepping schemes and clarify the relation between time-step restrictions and mesh parameters required to maintain non-negativity [2]. Numerical experiments for real-world models illustrate and validate the theoretical results.

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