Multiplicative Lie Crossed Modules of Multiplicative Lie Algebras and Their Isoclinism

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Multiplicative Lie algebras were introduced by G. J. Ellis in the study of universal commutator identities and the Ellis conjecture concerning higher weight commutators. These algebraic structures generalize classical Lie algebras and provide important connections between group theory, homotopy theory, and non-abelian algebraic systems. In recent years, multiplicative Lie algebras have attracted considerable attention due to their applications in commutator theory, homological algebra, and categorical algebra.

In this work, we introduce and study multiplicative Lie crossed modules in the category of multiplicative Lie algebras. This notion extends the classical crossed modules introduced by Whitehead to multiplicative Lie algebra structures equipped with compatible actions and multiplicative Lie products. A multiplicative Lie crossed module consists of a multiplicative Lie algebra homomorphism together with compatible actions satisfying suitable crossed module identities. We develop the basic theory of these crossed modules and investigate several structural properties associated with them.

Examples arising from ideals, central extensions, and self-actions of multiplicative Lie algebras are constructed to illustrate the theory. Further, notions such as crossed submodules, quotient crossed modules, centers, and commutator crossed submodules are introduced and studied in detail. In particular, it is shown that the commutator crossed submodule forms an ideal in the multiplicative Lie crossed module setting. Pullback constructions of crossed modules are also discussed, and the compatibility of induced actions is established.

Motivated by Hall's notion of isoclinism for groups and Lie algebras, we define isoclinism of multiplicative Lie crossed modules using suitable multiplicative Lie commutator maps and compatible isomorphisms. Several properties of isoclinic crossed modules are obtained. In particular, we prove that Lie aspherical and simply Lie connected multiplicative Lie crossed modules preserve the isoclinism of the associated multiplicative Lie algebras. These results provide a new structural approach for studying multiplicative Lie crossed modules and their algebraic behavior.

The present work contributes to the structural theory of multiplicative Lie algebras and opens directions for further research on classification problems, tensor products, homological aspects, and topological multiplicative Lie algebra structures.