Direct Product of Directed Graphs of Algebraic Length 1

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Let $\mathbb{A}$ and $\mathbb{C}$ be finite directed graphs. Then $\mathbb{A}^{\mathbb{C}}$ denotes the directed graph with vertex set A^C (the set of functions from C to A), and edges defined as follows: for $f, g \in A^C$, $f \to g$ in $\mathbb{A}^{\mathbb{C}}$ iff $f(c_1) \to g(c_2)$ holds in $\mathbb{A}$ whenever $c_1 \to c_2$ holds in $\mathbb{C}$. We then call $\mathbb{C}$ injective if $\mathbb{A}^{\mathbb{C}} \cong \mathbb{B}^{\mathbb{C}}$ implies $\mathbb{A} \cong \mathbb{B}$ for arbitrary finite directed graphs $\mathbb{A}, \mathbb{B}$.

In [1] László Lovász proved that a directed graph which has n vertices and contains a loop in every vertex but no other edges is injective. A variant of this injectivity problem for posets was solved by Ralph McKenzie in [2].

Examples of non-injective directed graphs are the bipartite ones and their natural generalizations, directed graphs which have algebraic length greater than 1, that is, there exists a $k > 1$ such that the directed cycle of length k is a homomorphic image of it. We do not know any other connected examples of non-injective graphs, so the question arises: Could directed graphs of algebraic length 1 and injective ones be the same among finite, connected directed graphs?

The class of injective directed graphs is closed under direct product, so we examined if the direct product of connected directed graphs of algebraic length 1 also has a component of the same algebraic length. This problem was set as a task at the Schweitzer competition in 2025.

[1] L. Lovász, Operations with structures, *Acta Mathematica Academiae Scientiarum Hungaricae* **18** (1967), 321–328.

[2] R. McKenzie, Arithmetic of Finite Ordered Sets: Cancellation of Exponents, II., *Order* **17** (2000), 309–332.