Approximation Error Analysis via Gram Matrices in Hilbert Spaces with Applications to Kernel Methods

Bita Khalife
Yasouj University, Yasouj, Iran

In this work, we study the role of Gram matrices in approximation problems in Hilbert spaces. Given a finite collection of linearly independent vectors, the associated Gram matrix is used to analyze orthogonal projections and approximation errors. The projection of an arbitrary element onto a finite-dimensional subspace generated by these vectors is considered. Using the Gram matrix structure, an expression for the approximation error is derived, and the influence of matrix conditioning on numerical stability and approximation quality is discussed. Connections with kernel methods in machine learning are also presented, where Gram matrices naturally arise as kernel matrices in reproducing kernel Hilbert spaces (RKHS). This interpretation provides a geometric framework for understanding approximation behavior in learning algorithms and large-scale data analysis. The results demonstrate that Gram matrix techniques provide a natural connection between classical functional analysis and modern machine learning applications. A simple numerical example illustrating the approximation framework will also be presented.