Approximation Error Analysis via Gram Matrices in Hilbert Spaces with Applications to Kernel Methods

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Gram matrices constitute a fundamental tool in the study of finite-dimensional approximations in Hilbert spaces. In this paper, we investigate the relationship between Gram matrix representations and orthogonal projection errors, establishing explicit error formulas and spectral bounds that characterize approximation accuracy and stability. In this work, we investigate finite-dimensional approximation problems in Hilbert spaces through the framework of Gram matrices. Let V be a finite-dimensional subspace generated by a collection of linearly independent vectors. We derive an explicit representation of the orthogonal projection error of an arbitrary element onto V in terms of the inverse Gram matrix associated with the generating system. Using spectral properties of Gram matrices, we establish upper and lower bounds for the approximation error and analyze the influence of matrix conditioning on the stability and accuracy of the approximation process. These estimates provide a quantitative description of how the geometric structure of the generating vectors affects approximation quality in Hilbert spaces. Furthermore, the developed framework is extended to reproducing kernel Hilbert spaces (RKHS), where Gram matrices naturally arise as kernel matrices. This connection enables the interpretation of the obtained results in the context of kernel-based machine learning algorithms and data approximation methods. The presented approach highlights the interplay between classical approximation theory, spectral analysis, and modern kernel methods. The results provide a unified perspective linking Hilbert space geometry, approximation error analysis, and machine learning applications. Several illustrative examples are discussed to demonstrate the applicability of the proposed framework.

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