One-Dimensional Strong Affine Representations of the Polycyclic Monoids

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The *polycyclic monoid* $\mathcal{P}_n$ is a monoid with zero given by the presentation

$$\mathcal{P}_n = \langle a_0, \ldots, a_{n-1}, a_0^{-1}, \ldots, a_{n-1}^{-1} : a_i^{-1} a_i = 1 \text{ and } a_i^{-1} a_j = 0, i \neq j \rangle.$$

Let $D = (d_0, d_1, \ldots, d_{n-1})$ be a complete system of residues modulo n and let us consider the functions $f_i \colon \mathbb{Z} \to \mathbb{Z}, x \mapsto nx + d_i$ $(i = 0, 1, \ldots, n-1)$. These functions give rise to a so-called one-dimensional strong affine representation of the polycyclic monoid $\mathcal{P}_n$.

This representation can be visualized by an edge-labeled directed graph: we draw an arrow with label i from x to $f_i(x)$ for each $x \in \mathbb{Z}$ and $i \in \{0, 1, \ldots, n-1\}$. Every connected component of such a graph contains exactly one cycle that can be uniquely identified by the word obtained by recording the labels along the edges of the cycle. It can be shown that the set of words corresponding to the cycles determine the edge-labeled graph up to isomorphism as well as the representation up to equivalence.

Our main problem is describing the set of words corresponding to a one-dimensional strong affine representation of $\mathcal{P}_n$. It has been proven that every finite set of words can be extended to obtain a set of words describing a representation, and we have previously shown how to do it if the system of residues is an arithmetic sequence starting with zero.

This talk will focus on a new result, where we have found a complete characterization for the sets of words induced by arbitrary arithmetic sequences (i.e., if $D = (c, c+h, c+2h, \ldots, c+(n-1)h)$ for some positive integers c and h where h is relatively prime to n), and that these sets (together with the empty set) form a closure system in the set of words over $\{0, 1, \ldots, n-1\}$. Furthermore, we will provide a method for obtaining the closure of an arbitrary set of words.

This is a joint work with Tamás Waldhauser.