Iteration of Multiplication and Ceiling Function on the Rationals

Tamás Waldhauser, **István Zalán Ruff**
University of Szeged, Hungary

We examine the properties of the operation $x \circ y = x\lceil y \rceil$. We present a handful of conjectures about the topic. The most important of them is that the set $\langle q \rangle_\circ$ generated from q by the operation $\circ$ contains an integer. Using the sequence $\lceil q \rceil, \lceil q\lceil q \rceil \rceil, \lceil \lceil q\lceil q \rceil \rceil \rceil, \ldots$, Chen's estimate for the lowest primitive root modulo p^2 and computer-aided testing, we proved the conjecture for values of q for which the numerator and denominator are close enough. The map $\tilde{g}_q(x) = \lceil qx \rceil$ is a conjugate map of $f_q(x) = q \circ x$. We describe results of Lagarias and Sloane regarding the exceptional set of $\tilde{g}_q$, which is the set containing integers on which the iteration of $\tilde{g}_q$ does not produce a number divisible by the denominator of q. We describe the exceptional set for $q = \frac{1}{l}$, which is an example for the exceptional set being infinite. Finally, we show that Lagarias's conjecture about the exceptional set being finite for $|q| > 1$ implies that the exact exceptional sets can be given for these values of q.

[1] J. C. Lagarias, N. J. A. Sloane, Approximate squaring, *Experiment. Math.* **13** (2004)

[2] Bo Chen, *Explicit upper bound on the least primitive root modulo* p^2, *Int J. Number Theory* **18** (2022)