Ω-Vector Spaces

Patricia Ferrero Plaza
University of Oviedo,Spain

The topic of this thesis belongs to the field of lattice valued (fuzzy) algebraic structures equipped with generalized equality relation, replacing the classical equality. A basic source for this research is the theory of Ω-sets. These structures appeared in 1979., by M.P. Fourman and D.S. Scott. An Ω-set is a nonempty set A equipped with an Ω-valued equality E, with truth-values in a particular complete lattice. Another source is the concept of algebras with a fuzzy equality, introduced by Bělohlávek and Vychodil. On this basis Ω-algebras are introduced by B. Šešelja and A. Tepavčević as algebras with generalized equalities.

An Ω-algebra is an algebraic structure equipped with the mentioned Ω-equality, being a particular map from the square of the algebra into a complete lattice. Classical identities are fulfilled as special lattice formulas. The quotient structures obtained by the cut of this generalized equality are classical algebras of the same type fulfilling these identities in the classical sense.

Our aim is to introduce and investigate the new structure in this framework, namely Ω-vector spaces. Our basic definition is as follows.

Let $\overline{\mathcal{G}} = (\mathcal{G},\ E)$ be an Ω-group ($\mathcal{G} = (G, +)$ is the underlying groupoid) and let $\mathcal{P} = (P, +, \cdot, , \iota)$ be a field. We assume the existence of a map $P \times G \to G$ denoted by the superposition, fulfilling

$E(a, b) \leqslant E(\alpha a, \alpha b)$, $a, b \in G$, $\alpha \in P$.

Under these conditions, we say that $\overline{\mathcal{G}}$ is an **Ω-vector space over** $\mathcal{P}$ if the following known formulas hold: for $a, b \in G$, $\alpha, \beta \in P$ and ι - the unit under $\cdot$ in P,

$(i)\ \alpha(a + b) = \alpha a + \alpha b$; $(ii)\ (\alpha + \beta)a = \alpha a + \beta a$; $(iii)\ \alpha(\beta a) = (\alpha \cdot \beta)a$; $(iv)\ \iota a = a$.

The above formulas are valid in $\overline{\mathcal{G}}$ if and only if for all $a, b \in G$, $\alpha, \beta \in P$ and ι - the unit under $\cdot$ in P, the following lattice-theoretic formulas hold, respectively:

$(i')\ \ \mu(a) \wedge \mu(b) \leqslant E(\alpha(a + b), \alpha a + \alpha b)$; $(ii')\ \ \mu(a) \leqslant E((\alpha + \beta)a, \alpha a + \beta a)$;
$(iii')\ \ \mu(a) \leqslant E(\alpha(\beta a), (\alpha \cdot \beta)a)$; $(iv')\ \ \mu(a) \leqslant E(\iota a, a)$.

Our aim is to investigate the basic properties of Ω-vector spaces and to show their applications in various aspects of linear algebra.

If $(\mathcal{G},\ E)$ is an Ω-vector space over a field $\mathcal{P}$, then the map $\mu : G \to \Omega$ such that $\mu(x) = E(x, x)$ is the *domain* of $\overline{\mathcal{G}}$. If $p \in \Omega$, then the *p-cuts* (cuts) of μ and E are subsets of G and G^2 respectively, defined by:

$\mu_p := \{x \in G \mid \mu(x) \geqslant p\}$; $E_p := \{(x, y) \in G^2 \mid E(x, y) \geqslant p\}$.

We prove that μ_p *is a subgroupoid of* G and E_p *is a congruence relation on* μ_p.

Theorem 1. *Let $(\mathcal{G}, E)$ be an Ω-group, where $(G, \cdot)$ is a groupoid and Ω a complete lattice. Let also $(P, +, \cdot)$ be a field. Then $(\mathcal{G}, E)$ is an Ω-vector space over P if and only if for every $p \in \Omega$, μ_p/E_p is a classical vector space over P.*

[1] B. Šešelja, A. Tepavčević, Ω-algebras, *Proceedings of ALHawai'i 2018, A conference in honor of Ralph Freese, William Lampe, and J.B. Nation*, (2018) 96–106.

[2] J. Jimenez, M.L. Serrano, B. Šešelja, A. Tepavčević, Ω-rings, *Fuzzy sets syst.* **455** (2023), 183–197.