Existence of a Homoclinic Solution for a Delay Differential Equation

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We consider the delay differential equation

$$y'(t) = -ay(t) + b \begin{cases} y^2(t-1) & \text{if } y(t-1) \in [0,1) \\ 0 & \text{if } y(t-1) \geq 1 \end{cases} \tag{1}$$

with $b > a > 0$. Equation (1) is the limit version (as $n \to \infty$) of the Mackey-Glass type equation $x'(t) = -ax(t) + bx^2(t-1)/[1 + x^n(t-1)]$. Considering the local 1-dimensional fast unstable manifold of the equilibrium point $\xi_* = \frac{a}{b}$ we get a solution $y : \mathbb{R} \to \mathbb{R}$ of Equation (1) such that

$$\lim_{t \to -\infty} y(t) = \xi_*, \quad y(0) = 1, \quad y(s) > 1, \text{ for } s \in [-1, 0).$$

If $\lim\limits_{t \to +\infty} y(t) = \xi_*$ then the solution y of Equation (1) is homoclinic to ξ_*. The transform $u(t) = by(t) - a$ leads to the equation

$$u'(t) = -au(t) + 2au(t-1) + u^2(t-1). \tag{2}$$

There is a unique $b^* > a$ so that $u^* : [-1, \infty) \to \mathbb{R}$ with $u^*(s) = b^* e^{-a(s+1)} - a$, $-1 \leq s \leq 0$, oscillates. For $a \in \left(0, \dfrac{5\pi}{3\sqrt{3}}\right)$, we construct a $\rho = \rho(a) > 0$ such that

$$B_\rho = \{\varphi \in C([-1, 0], \mathbb{R}) : ||\varphi|| < \rho\}$$

does not contain periodic orbits. This step requires a careful choice of the exponential dichotomy constants at the equilibrium $u = 0$. For $a \in (0, \widehat{a}]$, $\widehat{a} = 1.64$, a computer-assisted estimation of u^* guarantees $u^*_t \in B_\rho$ for all large t. Consequently, for any $a \in (0, \widehat{a}]$, there is a unique $b^* = b^*(a)$ such that Equation (1) with $b = b^*$ has a solution y satisfying $\lim\limits_{|t| \to +\infty} y(t) = \xi_*$.

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