Non-Diagonal Critical Central Secctions of the Cube

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We study the $(n-1)$-dimensional volume of central hyperplane sections of the n-dimensional cube Q_n, that is, sections of the form $Q_n \cap \mathbf{v}^\perp$. Global extrema of this functional were described by Hadwiger [3] and Ball [2] in the last century. The volume of central section is minimal and maximal if its normal vector is parallel to the main diagonal of a 1- and a 2-dimensional face of Q_n, respectively. More generally, a central section is called k-diagonal if its normal vector is parallel to the main diagonal of a k-dimensional face of Q_n. Our first result provides an alternative, simpler argument for proving that the volume of the section perpendicular to the main diagonal of the cube, i.e. n-diagonal section is strictly locally maximal for every $n \geq 4$, which was shown before by Pournin [6].

Further investigation needs study of critical points of central section function on the unit sphere S^{d-1}. These are referred to as critical directions, which were characterized by Ivanov and Tsiutsiurupa [4] and, independently, the first named author [1]. It has been widely believed that all critical sections of Q_n need to be diagonal. This has been verified for $n = 2, 3$, but disproved for $n = 4$ in [1]. Our second result generalized this phenomenon to higher dimensions: we prove that non-diagonal critical central sections of Q_n exist in all dimensions at least 4.

The crux of both proofs is an estimate on the rate of decay of the Laplace-Pólya integral $J_n(r) = \frac{1}{\pi}\int_{-\infty}^{\infty} \operatorname{sinc}^n t \cdot \cos(rt)\,\mathrm{d}t$ that is achieved by combinatorial means. This also yields improved bounds for Eulerian numbers of the first kind, which were investigateed by Lesieur and Nicolas [5].

This is a joint work with Gergely Ambrus from University of Szeged and Alfréd Rényi Institute of Mathematics.

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