On Group Algebras with Metabelian Group of Units

István Mozgovoj
Ferenc Rakóczi II Transcarpathian Hungarian College of Higher Education, Berehove, Ukraine

Let F be a field of characteristic p, G a group and denote by $U(FG)$ the group of units of the group algebra FG. In 1991 Shalev [1] gave a necessary and sufficient condition for $U(FG)$ to be metabelian (by a metabelian group we mean a group with an abelian commutator subgroup), provided that $p > 2$ and G is finite. A few years later Kurdics [2] and, independently, Coleman and Sandling [3] did the same for the case $p = 2$. (It's important to note, that in theory of group rings the $p = 2$ case is nearly always exceptional and require a different treatment compared to the $p > 2$ case.) The restriction for the order of G was relaxed by Catino and Spinelli [4], such that G may be torsion for odd p, and nilpotent torsion for $p = 2$. Recently Juhász and Spinelli [5] have handled the non-torsion case for odd p.

In this work, we take some steps to this direction for the $p = 2$ case. In particular, we find all non-nilpotent groups G, for which $U(FG)$ is metabelian. We do not impose any restriction to the order of G, however, when G is not torsion, we must require the group algebra FG to be modular (that is, G contains an element of order 2). Namely, we prove the following theorems.

Theorem 1. *Let G be a non-abelian torsion group and F a field of characteristic 2. Then $U(FG)$ is metabelian, if and only if, one of the following statements holds:*

1. *The commutator subgroup G' of G is a central elementary abelian group of order 2 or 4;*
2. *F is the field of two elements, and G is an extension of an elementary abelian 3-group H by the group $\langle b \rangle$ of order 2 with $b^{-1}ab = a^{-1}$ for every $a \in H$.*

Theorem 2. *Let G be a non-torsion group and F a field of characteristic 2. If FG is modular and $U(FG)$ is metabelian, than G is nilpotent.*

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