Probability Equivalent Level of Value at Risk and Higher-Order Expected Shortfalls

Mátyás Barczy, **Fanni K. Nedényi**, László Sütő
ELKH-SZTE Analysis and Stochastics Research Group, University of Szeged, Szeged, Hungary

The Fundamental Review of the Trading Book (FRTB) was introduced by the Basel Committee on Banking Supervision in the years following the Global Financial Crisis of 2007-2009. FRTB is expected to make a complete revision of the approach to calculating risk-based capital requirements for investments. Currently, banks are required to report their risk calculated as the 99% Value at Risk. Consider a random variable X which is the loss on a certain portfolio. Then, the Value at Risk (VaR) of X at a level $p \in [0,1]$ is defined by

$$\mathtt{VaR}_X(p) := \inf\{x \in \mathbb{R} : F_X(x) \geqslant p\},$$

with the convention $\inf \emptyset := \infty$. Then, roughly speaking, $\mathtt{VaR}_X(0.99)$ is the loss that is likely to be exceeded only 1% of the time. While VaR is widely used and easy to compute, it has no information on the magnitude of the biggest 1% of losses. Moreover, it is not a coherent risk measure. Indeed, it is not subadditive, which means that VaR of a portfolio can be higher than the sum of VaRs of the individual assets in the portfolio.

By FRTB, banks will soon have to switch to Expected Shortfall (ES) at the level 0.975 instead of VaR at the level 0.99 for the bank-wide internal models to determine market risk capital requirements. We are interested in studying the possibility to switch to higher-order Expected Shortfalls instead. Let X be a random variable such that $X \in L^1$, and let $n \in \mathbb{N}$. The n^{th}-order ES of X at a level $p \in [0,1)$ is defined by

$$\mathtt{ES}_{X,n}(p) := \frac{n}{1-p} \int_p^1 \left(\frac{s-p}{1-p}\right)^{n-1} \mathtt{VaR}_X(s)\,\mathrm{d}s.$$

This is the classical ES when $n = 1$.

Motivated by the work of Li and Wang [2] for ES, we define the probability equivalent level of VaR and n^{th}-order ES, called $\mathtt{PELVE}_n$. For an integrable random variable X and $\varepsilon \in (0,1)$, the $\mathtt{PELVE}_n$ of X at the level ε is the infimum of those values $c \in [1, \frac{1}{\varepsilon}]$ for which $\mathtt{ES}_{X,n}(1-c\varepsilon) \leqslant \mathtt{VaR}_X(1-\varepsilon)$. One can see that the level $\varepsilon = 0.01$ corresponds to the replacement of VaR at the level 0.99 with n^{th}-order ES at some appropriate level. We study the properties of $\mathtt{PELVE}_n$ and calculate $\mathtt{PELVE}_2$ for some important distributions including ones with heavy tail. Moreover, for $\mathtt{PELVE}_2$, we present some simulation results along with real data analysis.

[1] Barczy, M., K. Nedényi, F. and Sütő, L. (2022). Probability equivalent level of Value at Risk and higher-order Expected Shortfalls.
ArXiv: `https://arxiv.org/abs/2202.09770`

[2] Li, H. and Wang, R. (2019). PELVE: Probability Equivalent Level of VaR and ES. To appear in *Journal of Econometrics. Available also at SSRN.* **30(2)** 325–341.
URL: `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3489566`