On the Automorphism Group of the Substructure Ordering of Finite Directed Graphs

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We investigate the automorphisms, i. e. symmetries, of a concrete partially ordered set, shortly poset. The elements of the poset in question are finite directed graphs and what gives the order relation between them is some sort of containment. Let $\mathcal{D}$ denote the set of (isomorphism types of) finite directed graphs, and, for $G, G' \in \mathcal{D}$, let $G \sqsubseteq G'$ denote the fact that G is (isomorphic to) a spanned subgraph of G'. The partial order we are investigating is $(\mathcal{D}; \sqsubseteq)$. In [1], the second author formulated a conjecture for the automorphism group of $(\mathcal{D}; \sqsubseteq)$. This conjecture was not at all placed on a firm footing. In this work, though still unable to prove it, we give it a firm footing with the use of some computer calculations.

It is easy to see that $(\mathcal{D}; \sqsubseteq)$ is a graded poset, that is it splits into levels the natural way. On the n-th level, we find the digraphs having n vertices. In [1], it is shown that the action of an automorphism on the first twelve levels determines it wholly. At first sight, this seems to reduce our infinite problem (of finding all the automorphisms) to a finite one. This is where computer calculations come into play as finite calculations can be done with a fast enough computer. The problem is that we can't even get close to the computational capacity that would be needed here. Nevertheless, we show that on the first few levels the automorphism group behaves according to the conjecture.

Let $\operatorname{Aut} P$ denote the automorphism group of the poset P. For a graded poset P, let P_n denote the subposet of the first n levels. It is clear that P_n is invariant under $\operatorname{Aut} P$. It turns out that we cannot really expect $\operatorname{Aut} \mathcal{D} \cong \operatorname{Aut} \mathcal{D}_n$ for any concrete n. For example we show that

$$|\operatorname{Aut} \mathcal{D}_3| \approx 1.67 \cdot 10^{13}, \tag{1}$$

which is way too big as the group of the conjecture has 768 elements. This forces us to do the following. Let $\varphi|_n$ denote the restriction of $\varphi \in \operatorname{Aut} P$ to P_n. Furthermore, let $\operatorname{Aut}_n P$ denote $\{\varphi|_n \in \operatorname{Aut} P_n : \varphi \in \operatorname{Aut} P\}$, which is clearly a subgroup of $\operatorname{Aut} P_n$. With these notations, what we show is that, in sharp contrast to (1), $\operatorname{Aut}_3 \mathcal{D}_4$ is isomorphic to the 768-element group of our conjecture, meaning that the automorphism group works on the first three levels as expected by the conjecture.

[1] Á. Kunos, Definability in the substructure ordering of finite directed graphs, *Order* **38** (2021), 401–420.