Characterization of Minimally Tough Chordal Graphs with Small Toughness

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Let t be a real number. A graph is called *t-tough* if the removal of any vertex set S that disconnects the graph leaves at most $|S|/t$ components. The toughness of a graph is the largest t for which the graph is t-tough. A graph is minimally t-tough if the toughness of the graph is t and the deletion of any edge from the graph decreases the toughness. A graph is *chordal* if it does not contain an induced cycle of length at least 4. We will call a graph a *TT-graph* if it can be obtained from a tree of maximum degree $\Delta \geq 3$ by removing some (or all) of its vertices with degree 3 whose neighbors have degree Δ, and joining these neighbors with triangle.

Kriesell's conjectured [1] that every minimally 1-tough graph has a vertex of degree 2. This conjecture can be naturally generalized: every minimally t-tough graph has a vertex of degree $\lceil 2t \rceil$. Gyula Y. Katona and Kitti Varga [3], showed that the conjecture is true for chordal graphs when $1/2 < t \leq 1$.

In this paper we show that the Generalized Kriesell's Conjecture for chordal graphs with toughness $\leq 1/2$ by giving a characterization of such graphs. We show that for $t \leq 1/2$ a chordal graph is minimally t-tough if and only if it is a TT-graph. As a corollary, a characterization of minimally t-tough interval graphs is obtained for $t \leq 1/2$, as well.

One of the main tools in the proof is the well known representation of chordal graphs using clique trees [2]. The other important tool is general necessary and sufficient condition for minimally tough graphs given in [4].

[1] Mathias Kriesell, *In: Ed. T. Kaiser, Problems from the Workshop on dominating cycles*, Hájek, Czech Republic, 2003.
`http://iti.zcu.cz/history/2003/Hajek/problems/hajek-problems.ps`

[2] C. G. Lekkerkerker and J. Ch. Boland, *Representation of a finite graph by a set of intervals on the real line*, Fund. Math., 51 (1962), pp. 45–64.

[3] Gyula Y. Katona, Kitti Varga, *Minimally toughness in special graph classes,* arXiv:1802.00055 [math.CO] 2018.

[4] Clément Dallard, Blas Fernández, Gyula Y. Katona, Martin Milanič, *Conditions for minimally tough graphs*, manuscript 2021.