Mal'tsev Conditions and Graph Powers

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Let $\mathcal{I}$ be a finite set of identities over any signature. Then $\mathcal{I}$ induces a *strong Mal'tsev class* denoted by $\mathfrak{M}(\mathcal{I})$: an algebra is in this class precisely if it has terms that satisfy $\mathcal{I}$. For example, if

$$\mathcal{I} = \{a+b \approx b+a, (a+b)+c \approx a+(b+c), a-a \approx 0, a+0 \approx a, a-b \approx a+(0-b), \\ a \star (b+c) \approx a \star b + a \star c, (b+c) \star a \approx b \star a + c \star a, (a \star b) \star c \approx a \star (b \star c)\},$$

and

$$\mathcal{J} = \{a+b \approx b+a, (a+b)+c \approx a+(b+c), a-a \approx 0, a+0 \approx a, a-b \approx a+(0-b), \\ a\star(b+c) \approx a\star b+a\star c, (b+c)\star a \approx b\star a+c\star a, a\star a \approx 0, (a\star b)\star c+b\star(c\star a)+c\star(a\star b)\},$$

then an algebra is in $\mathfrak{M}(\mathcal{I})$ and $\mathfrak{M}(\mathcal{J})$ if it has ring terms and Lie-ring terms, respectively. Somewhat surprisingly, these classes are the same: a ring becomes a Lie-ring if we change the multiplication to the commutator operation defined by $[a,b] := ab - ba$, also, a Lie-ring becomes a ring if we change the Lie-bracket into the the operation defined by $ab := 0$.

It can be very hard to decide whether two sets of identities generate the same strong Mal'tsev class, or whether they contain each other. A related question is whether such a class $\mathfrak{K}$ is *prime*: is it true that if $\mathfrak{K}$ contains $\mathfrak{L}_1 \cap \mathfrak{L}_2$ then it contains either $\mathfrak{L}_1$ or $\mathfrak{L}_2$?

We have answered this question affirmatively in the case when $\mathfrak{K}$ is the class of algebras that generate a congruence permutable variety (induced by the identities $m(x,x,y) \approx m(y,x,x) \approx y$). While this is a purely algebraic result, the proof uses exponentiation of graphs. For (directed or undirected) graphs $\mathbb{G}$ and $\mathbb{H}$ the vertices of the graph $\mathbb{G}^{\mathbb{H}}$ are the mappings $H \to G$, and the edge relation is defined by

$$f_1 \to f_2 \Leftrightarrow \forall\, h_1 \to h_2 : \; f_1(h_1) \to f_2(h_2).$$

The main combinatoric ingredient to our algebraic result is the following simple statement:

Theorem 1. *If $\mathbb{G}$ and $\mathbb{H}$ are graphs such that neither if complete, but each has a vertex that is a neighbor of any vertex (including itself), then there is a nonempty graph $\mathbb{K}$ so that $\mathbb{G}^{\mathbb{K}}$ and $\mathbb{H}^{\mathbb{K}}$ are isomorphic.*