Representations of reciprocals of Lucas sequences

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In 1953 Stancliff [2] noted an interesting property of the Fibonacci number $F_{11} = 89$. One has that

$$\frac{1}{89} = \frac{F_0}{10} + \frac{F_1}{10^2} + \frac{F_2}{10^3} + \frac{F_3}{10^4} + \frac{F_4}{10^5} + \frac{F_5}{10^6} + \dots.$$

De Weger [1] determined all $x \geq 2$ for which $\frac{1}{F_n} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{x^k}$. The solutions are as follows

$$\frac{1}{F_1} = \frac{1}{F_2} = \frac{1}{1} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{2^k}, \qquad \frac{1}{F_5} = \frac{1}{5} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{3^k},$$
$$\frac{1}{F_{10}} = \frac{1}{55} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{8^k}, \qquad \frac{1}{F_{11}} = \frac{1}{89} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{10^k}.$$

In 2014 Tengely [3] extended the above result and obtained e.g. $\frac{1}{U_{10}} = \frac{1}{416020} = \sum_{k=0}^{\infty} \frac{U_k}{647^{k+1}}$, where $U_0 = 0, U_1 = 1$ and $U_n = 4U_{n-1} + U_{n-2}, n \geq 2$. In this talk we focus on the following problems. We deal with equations of the form

$$\frac{1}{U_n(P_2, Q_2)} = \sum_{k=0}^{\infty} \frac{U_k(P_1, Q_1)}{x^{k+1}},$$

for certain given pairs $(P_1, Q_1) \neq (P_2, Q_2)$ with $1 \leq P \leq 3$ and $Q = \pm 1$.
We also study equations of the form

$$\sum_{k=0}^{\infty} \frac{U_k(P, Q)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{R_k}{y^{k+1}},$$

where R_n is a ternary linear recurrence sequence of the form $R_0 = R_1 = 0, R_2 = 1$ and $R_n = CR_{n-1} + DR_{n-2} + ER_{n-3}$. We provide results related to equations

$$\sum_{k=0}^{\infty} \frac{U_k(P_1, Q_1)}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{V_k(P_2, Q_2)}{y^{k+1}}, \qquad \sum_{k=0}^{\infty} \frac{R_k}{x^{k+1}} = \sum_{k=0}^{\infty} \frac{T_k}{y^{k+1}}$$

where $\{U_k\}, \{V_k\}$ are Lucas sequences and $\{R_k\}$ and $\{T_k\}$ are ternary linear recurrence sequences. This is a joint work with (Szabolcs Tengely from the University of Debrecen).

[1] B. M. M. de Weger, *A curious property of the eleventh Fibonacci number*, Rocky Mountain J. Math., 25 (1995), pp. 977–994.

[2] F. Stancliff, *A curious property of a_{ii}*, Scripta Math.,19:126, 1953.

[3] Sz. Tengely,On the Lucas sequence equation $\frac{1}{U_n} = \sum_{k=1}^{\infty} \frac{U_{k-1}}{x^k}$, *J. Period. Math. Hung.* **71(2)** (2014), 236-242.