Weakly infinitely divisible measures on some locally compact Abelian groups

Mátyás Barczy[∗,⋄] and **Gyula Pap**[∗]

∗ Faculty of Informatics, University of Debrecen, Pf. 12, H–4010 Debrecen, Hungary; E–mail: barczy@inf.unideb.hu (M. Barczy); papgy@inf.unideb.hu (G. Pap).

⋄ Corresponding author.

Abstract. On the torus group, on the group of p–adic integers and on the p–adic solenoid we give a construction of an arbitrary weakly infinitely divisible probability measure using a random element with values in a product of (possibly infinitely many) subgroups of $\mathbb{R}$. As a special case of our results, we have a new construction of the Haar measure on the p–adic solenoid.

2000 Mathematics Subject Classification. Primary: 60B15, Secondary: 22B99

Key words. Weakly infinitely divisible measure; torus group; group of p–adic integers; p–adic solenoid; Haar measure.

Running head. Weakly infinitely divisible measures on some LCA groups.

1 Introduction

Weakly infinitely divisible probability measures play a very important role in limit theorems of probability theory, see for example, the books of Parthasarathy [11], Heyer [9], the papers of Bingham [4], Yasuda [14], Barczy, Bendikov and Pap [2], and the Ph.D. theses of Gaiser [7], Telöken [12], Barczy [1]. They naturally arise as possible limits of triangular arrays described as follows.

Let G be a locally compact Abelian topological group having a countable basis of its topology. We also suppose that G has the T_0–property, that is, $\bigcap_{U\in\mathcal{N}_e} U = \{e\}$, where e denotes the identity element of G and $\mathcal{N}_e$ is the collection of all Borel neighbourhoods of e. (By a Borel neighbourhood U of e we mean a Borel subset of G for which there exists an open subset $\widetilde{U}$ of G such that $e \in \widetilde{U} \subset U$.) Let us consider a probability measure μ on G and let $\{X_{n,k} : n \in \mathbb{N},\ k = 1,\dots,K_n\}$ be an array of rowwise independent random elements with values in G satisfying the infinitesimality condition

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant K_n} \mathrm{P}(X_{n,k} \in G \setminus U) = 0, \qquad \forall\, U \in \mathcal{N}_e.$$

If the row sums $\sum_{k=1}^{K_n} X_{n,k}$ of such an array converge in distribution to μ then μ is necessarily weakly infinitely divisible, see, e.g., Parthasarathy [11, Chapter IV, Theorem 5.2]. Moreover, Parthasarathy [11, Chapter IV, Corollary 7.1] gives a representation of an arbitrary weakly infinitely divisible measure on G in terms of a Haar measure, a Dirac measure, a Gauss measure and a Poisson measure on G.

In this paper we consider the torus group, the group of p–adic integers and the p–adic solenoid. For these groups, we give a construction of an arbitrary weakly infinitely divisible measure using real random variables. For each of these three groups, the construction consists in:

(i) finding a group, say G_0, which is a product of (possibly infinitely many) subgroups of $\mathbb{R}$. (We furnish G_0 with the product topology. Note that G_0 is not necessarily locally compact.)

(ii) finding a continuous homomorphism $\varphi : G_0 \to G$ such that for each weakly infinitely divisible measure μ on G, there is a probability measure μ_0 on G_0 with the property $\mu_0(\varphi^{-1}(B)) = \mu(B)$ for all Borel subsets B of G. (The probability measure μ_0 on G_0 will be given as the distribution of an appropriate random element with values in G_0.)

Since φ is a homomorphism, the building blocks of μ (Haar measure, Dirac measure, Gauss measure and Poisson measure) can be handled separately.

We note that, as a special case of our results, we have a new construction of the Haar measure on the p-adic integers and the p-adic solenoid. Another kind of description of the Haar measure on the p-adic integers can also be found in Hewitt and Ross [8, p. 220]. One can find a construction of the Haar measure on the p-adic solenoid in Chistyakov [6, Section 3]. It is based on Hausdorff measures and rather sophisticated, while our simpler construction (Theorem 5.1) is based on a probabilistic method and reflects the structure of the p-adic solenoid.

2 Parametrization of weakly infinitely divisible measures

Let $\mathbb{N}$ and $\mathbb{Z}_+$ denote the sets of positive and of nonnegative integers, respectively. The expression "a measure μ on G" means a measure μ on the σ–algebra of Borel subsets of G. The Dirac measure at a point $x \in G$ will be denoted by δ_x.

2.1 Definition. A probability measure μ on G is called *infinitely divisible* if for all $n \in \mathbb{N}$, there exist a probability measure μ_n on G such that $\mu = \mu_n^{*n}$. The collection of all infinitely divisible measures on G will be denoted by $\mathcal{I}(G)$. A probability measure μ on G is called *weakly infinitely divisible* if for all $n \in \mathbb{N}$, there exist a probability measure μ_n on G and an element $x_n \in G$ such that $\mu = \mu_n^{*n} * \delta_{x_n}$. The collection of all weakly infinitely divisible measures on G will be denoted by $\mathcal{I}_{\mathrm{w}}(G)$.

Note that $\mathcal{I}(G) \subset \mathcal{I}_{\mathrm{w}}(G)$, but in general $\mathcal{I}(G) \neq \mathcal{I}_{\mathrm{w}}(G)$. Clearly, $\mathcal{I}(G) = \mathcal{I}_{\mathrm{w}}(G)$ if all the Dirac measures on G are infinitely divisible. In case of the torus and the p–adic solenoid, $\mathcal{I}(G) = \mathcal{I}_{\mathrm{w}}(G)$, see Sections 3 and 5; and in case of the p–adic integers, $\mathcal{I}(G) \neq \mathcal{I}_{\mathrm{w}}(G)$, see the example in Section 4. We also remark that Parthasarathy [11] and Yasuda [14] call weakly infinitely divisible measures on G infinitely divisible measures.

We recall the building blocks of weakly infinitely divisible measures. The main tool for their description is Fourier transformation. A function $\chi : G \to \mathbb{T}$ is said to be a character of G if it is a continuous homomorphism, where $\mathbb{T}$ is the topological group of complex numbers $\{e^{ix} : -\pi \leqslant x < \pi\}$ under multiplication (for more details on $\mathbb{T}$, see Section 3). The group of all characters of G is called the character group of G and is denoted by $\widehat{G}$. The character group $\widehat{G}$ of G is also a locally compact Abelian T_0–topological group having a countable basis of its topology (see, e.g., Theorems 23.15 and 24.14 in Hewitt and Ross [8]). For every bounded measure μ on G, let $\widehat{\mu} : \widehat{G} \to \mathbb{C}$ be defined by

$$\widehat{\mu}(\chi) := \int_G \chi \, d\mu, \qquad \chi \in \widehat{G}.$$

This function $\widehat{\mu}$ is called the *Fourier transform* of μ. The usual properties of the Fourier transformation can be found, e.g., in Heyer [9, Theorem 1.3.8, Theorem 1.4.2], in Hewitt and Ross [8, Theorem 23.10] and in Parthasarathy [11, Chapter IV, Theorem 3.3].

If H is a compact subgroup of G then ω_H will denote the Haar measure on H (considered as a measure on G) normalized by the requirement $\omega_H(H) = 1$. The normalized Haar measures of compact subgroups of G are the only idempotents in the semigroup of probability measures on G (see, e.g., Wendel [13, Theorem 1]). For all $\chi \in \widehat{G}$,

$$\widehat{\omega}_H(\chi) = \begin{cases} 1 & \text{if } \chi(x) = 1 \text{ for all } x \in H, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

i.e., $\widehat{\omega}_H = \mathbb{1}_{H^\perp}$, where

$$H^\perp := \big\{\chi \in \widehat{G} : \chi(x) = 1 \text{ for all } x \in H\big\}$$

is the annihilator of H. Clearly $\omega_H \in \mathcal{I}_w(G)$, since $\omega_H * \omega_H = \omega_H$.

Obviously $\delta_x \in \mathcal{I}_w(G)$ for all $x \in G$.

A *quadratic form* on $\widehat{G}$ is a nonnegative continuous function $\psi : \widehat{G} \to \mathbb{R}_+$ such that

$$\psi(\chi_1\chi_2) + \psi(\chi_1\chi_2^{-1}) = 2(\psi(\chi_1) + \psi(\chi_2)) \qquad \text{for all } \chi_1, \chi_2 \in \widehat{G}.$$

The set of all quadratic forms on $\widehat{G}$ will be denoted by $q_+(\widehat{G})$. For a quadratic form $\psi \in q_+(\widehat{G})$, there exists a unique probability measure γ_ψ on G determined by

$$\widehat{\gamma}_\psi(\chi) = e^{-\psi(\chi)/2} \qquad \text{for all } \chi \in \widehat{G},$$

which is a symmetric Gauss measure (see, e.g., Theorem 5.2.8 in Heyer [9]). Obviously $\gamma_\psi \in \mathcal{I}_w(G)$, since $\gamma_\psi = \gamma_{\psi/n}^{*n}$ for all $n \in \mathbb{N}$.

For a bounded measure η on G, the *compound Poisson measure* $e(\eta)$ is the probability measure on G defined by

$$e(\eta) := e^{-\eta(G)} \left(\delta_e + \eta + \frac{\eta * \eta}{2!} + \frac{\eta * \eta * \eta}{3!} + \cdots\right),$$

where e is the identity element of G. The Fourier transform of a compound Poisson measure $e(\eta)$ is

$$(e(\eta))\widehat{\ }(\chi) = \exp\left\{\int_G (\chi(x) - 1)\, d\eta(x)\right\}, \qquad \chi \in \widehat{G}. \tag{2.2}$$

Obviously $\mathrm{e}(\eta) \in \mathcal{I}_{\mathrm{w}}(G)$, since $\mathrm{e}(\eta) = \big(\mathrm{e}(\eta/n)\big)^{*n}$ for all $n \in \mathbb{N}$.

In order to introduce generalized Poisson measures, we recall the notions of a local inner product and a Lévy measure.

2.2 Definition. A continuous function $g : G \times \widehat{G} \to \mathbb{R}$ is called a *local inner product* for G if

(i) for every compact subset C of $\widehat{G}$, there exists $U \in \mathcal{N}_e$ such that

$$\chi(x) = \mathrm{e}^{ig(x,\chi)} \qquad \text{for all } x \in U, \quad \chi \in C,$$

(ii) for all $x \in G$ and $\chi, \chi_1, \chi_2 \in \widehat{G}$,

$$g(x, \chi_1\chi_2) = g(x, \chi_1) + g(x, \chi_2), \qquad g(-x, \chi) = -g(x, \chi),$$

(iii) for every compact subset C of $\widehat{G}$,

$$\sup_{x\in G} \sup_{\chi\in C} |g(x, \chi)| < \infty, \qquad \lim_{x\to e} \sup_{\chi\in C} |g(x, \chi)| = 0.$$

Parthasarathy [11, Chapter IV, Lemma 5.3] proved the existence of a local inner product for an arbitrary locally compact Abelian T_0–topological group having a countable basis of its topology.

2.3 Definition. A measure η on G with values in $[0, +\infty]$ is said to be a *Lévy measure* if $\eta(\{e\}) = 0$, $\eta(G \setminus U) < \infty$ for all $U \in \mathcal{N}_e$, and $\int_G (1 - \operatorname{Re}\chi(x))\,\mathrm{d}\eta(x) < \infty$ for all $\chi \in \widehat{G}$. The set of all Lévy measures on G will be denoted by $\mathbb{L}(G)$.

We note that for all $\chi \in \widehat{G}$ there exists $U \in \mathcal{N}_e$ such that

$$\frac{1}{4} g(x, \chi)^2 \leqslant 1 - \operatorname{Re}\chi(x) \leqslant \frac{1}{2} g(x, \chi)^2, \qquad x \in U. \tag{2.3}$$

Thus the requirement $\int_G (1 - \operatorname{Re}\chi(x))\,\mathrm{d}\eta(x) < \infty$ can be replaced by the requirement that $\int_G g(x, \chi)^2\,\mathrm{d}\eta(x) < \infty$ for some (and then necessarily for any) local inner product g.

For a Lévy measure $\eta \in \mathbb{L}(G)$ and for a local inner product g for G, the *generalized Poisson measure* $\pi_{\eta,g}$ is the probability measure on G defined by

$$\widehat{\pi}_{\eta,g}(\chi) = \exp\left\{ \int_G \big(\chi(x) - 1 - ig(x, \chi)\big)\,\mathrm{d}\eta(x) \right\} \qquad \text{for all } \chi \in \widehat{G}$$

(see, e.g., Chapter IV, Theorem 7.1 in Parthasarathy [11]). Obviously $\pi_{\eta,g} \in \mathcal{I}_{\mathrm{w}}(G)$, since $\pi_{\eta,g} = \pi_{\eta/n,g}^{*n}$ for all $n \in \mathbb{N}$. Note that for a bounded measure η on G with $\eta(\{e\}) = 0$ we have $\eta \in \mathbb{L}(G)$ and $\mathrm{e}(\eta) = \pi_{\eta,g} * \delta_{m_g(\eta)}$, where the element $m_g(\eta) \in G$, called the *local mean* of η with respect to the local inner product g, is uniquely defined by

$$\chi(m_g(\eta)) = \exp\left\{ i \int_G g(x, \chi)\,\mathrm{d}\eta(x) \right\} \qquad \text{for all } \chi \in \widehat{G}.$$

(The existence of a unique local mean is guaranteed by Pontryagin's duality theorem.)

Let $\mathcal{P}(G)$ be the set of quadruplets (H, a, ψ, η), where H is a compact subgroup of G, $a \in G$, $\psi \in \mathrm{q}_+(\widehat{G})$ and $\eta \in \mathbb{L}(G)$. Parthasarathy [11, Chapter IV, Corollary 7.1] proved the following parametrization for weakly infinitely divisible measures on G.

2.4 Theorem. (Parthasarathy) *Let* g *be a fixed local inner product for* G. *If* $\mu \in \mathcal{I}_{\mathrm{w}}(G)$ *then there exists a quadruplet* $(H, a, \psi, \eta) \in \mathcal{P}(G)$ *such that*

$$\mu = \omega_H * \delta_a * \gamma_\psi * \pi_{\eta, g}. \tag{2.4}$$

Conversely, if $(H, a, \psi, \eta) \in \mathcal{P}(G)$ *then* $\omega_H * \delta_a * \gamma_\psi * \pi_{\eta, g} \in \mathcal{I}_{\mathrm{w}}(G)$.

In general, this parametrization is not one–to–one (see Parthasarathy [11, p.112, Remark 3]).

We say that $\mu \in \mathcal{I}_{\mathrm{w}}(G)$ has a non-degenerate idempotent factor if $\mu = \mu' * \nu$ for some probability measures μ' and ν such that ν is idempotent and $\nu \neq \delta_e$. Yasuda [14, Proposition 1] proved the following characterization of weakly infinitely divisible measures on G without non-degenerate idempotent factors, i.e., weakly infinitely divisible measures on G for which in the representation (2.4) the compact subgroup H is $\{e\}$.

2.5 Theorem. (Yasuda) *A probability measure* μ *on* G *is weakly infinitely divisible without non-degenerate idempotent factors if and only if there exist an element* $a \in G$ *and a triangular array* $\{\mu_{n,k} : n \in \mathbb{N}, k = 1, \ldots, K_n\}$ *of probability measures on* G *such that*

(i) *for every compact subset* C *of* $\widehat{G}$,

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant K_n} \sup_{\chi\in C} |\widehat{\mu}_{n,k}(\chi) - 1| = 0,$$

(ii) *for all* $\chi \in \widehat{G}$,

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{K_n} (1 - |\widehat{\mu}_{n,k}(\chi)|) < +\infty,$$

(iii) $\delta_a * \mathop{\Huge\ast}_{k=1}^{K_n} \mu_{n,k} \stackrel{\mathrm{w}}{\longrightarrow} \mu$ *as* $n \to \infty$, *where* $\stackrel{\mathrm{w}}{\longrightarrow}$ *means weak convergence and* $\mathop{\Huge\ast}_{k=1}^{K_n} \mu_{n,k}$ *denotes the convolution of* $\mu_{n,k}$, $k = 1, \ldots, K_n$.

We note that condition (i) of Theorem 2.5 is equivalent to the infinitesimality of the triangular array $\{\mu_{n,k} : n \in \mathbb{N}, k = 1, \ldots, K_n\}$, see, e.g., 5.1.12 in Heyer [9].

3 Weakly infinitely divisible measures on the torus

Consider the set $\mathbb{T} := \{\mathrm{e}^{ix} : -\pi \leqslant x < \pi\}$ of complex numbers under multiplication. This is a compact Abelian T_0–topological group having a countable basis of its topology, and it is called the 1–dimensional torus group. For elementary facts about $\mathbb{T}$ we refer to the monographs Hewitt and Ross [8], Heyer [9] and Hofmann and Morris [10]. The character group of $\mathbb{T}$ is $\widehat{\mathbb{T}} = \{\chi_\ell : \ell \in \mathbb{Z}\}$, where

$$\chi_\ell(y) := y^\ell, \qquad y \in \mathbb{T}, \quad \ell \in \mathbb{Z}.$$

Hence $\widehat{\mathbb{T}} \cong \mathbb{Z}$ (i.e., $\widehat{\mathbb{T}}$ and $\mathbb{Z}$ are topologically isomorphic). The compact subgroups of $\mathbb{T}$ are

$$H_r := \{\mathrm{e}^{2\pi i j/r} : j = 0, 1, \ldots, r-1\}, \qquad r \in \mathbb{N},$$

and $\mathbb{T}$ itself.

The set of all quadratic forms on $\widehat{\mathbb{T}} \cong \mathbb{Z}$ is $\mathrm{q}_+\big(\widehat{\mathbb{T}}\big) = \{\psi_b : b \in \mathbb{R}_+\}$, where

$$\psi_b(\chi_\ell) := b\ell^2, \qquad \ell \in \mathbb{Z}, \quad b \in \mathbb{R}_+.$$

Let us define the functions $\arg : \mathbb{T} \to [-\pi, \pi[$ and $h : \mathbb{R} \to \mathbb{R}$ by

$$\arg(\mathrm{e}^{ix}) := x, \qquad -\pi \leqslant x < \pi,$$

$$h(x) := \begin{cases} 0 & \text{if } x < -\pi \text{ or } x \geqslant \pi, \\ -x-\pi & \text{if } -\pi \leqslant x < -\pi/2, \\ x & \text{if } -\pi/2 \leqslant x < \pi/2, \\ -x+\pi & \text{if } \pi/2 \leqslant x < \pi. \end{cases}$$

A measure η on $\mathbb{T}$ with values in $[0, +\infty]$ is a Lévy measure if and only if $\eta(\{e\}) = 0$ and $\int_{\mathbb{T}} (\arg y)^2 \,\mathrm{d}\eta(y) < \infty$. The function $g_{\mathbb{T}} : \mathbb{T} \times \widehat{\mathbb{T}} \to \mathbb{R}$,

$$g_{\mathbb{T}}(y, \chi_\ell) := \ell h(\arg y), \qquad y \in \mathbb{T}, \quad \ell \in \mathbb{Z},$$

is a local inner product for $\mathbb{T}$.

Note that $\mathcal{I}(\mathbb{T}) = \mathcal{I}_{\mathrm{w}}(\mathbb{T})$, since $(\mathrm{e}^{ix/n})^n = \mathrm{e}^{ix}$, $x \in [-\pi, \pi)$, $n \in \mathbb{N}$.

Our aim is to show that for a weakly infinitely divisible measure μ on $\mathbb{T}$ there exist independent real random variables U and Z such that U is uniformly distributed on a suitable subset of $\mathbb{R}$, Z has an infinitely divisible distribution on $\mathbb{R}$, and $\mathrm{e}^{i(U+Z)} \stackrel{\mathcal{D}}{=} \mu$. We note that $\mathbb{R}$ is a locally compact Abelian T_0–topological group, its character group is $\widehat{\mathbb{R}} = \{\chi_y : y \in \mathbb{R}\}$, where $\chi_y(x) := \mathrm{e}^{iyx}$. The function $g_{\mathbb{R}} : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{R}$, defined by $g_{\mathbb{R}}(x, \chi_y) := yh(x)$, is a local inner product for $\mathbb{R}$.

3.1 Theorem. *If* $(H, a, \psi_b, \eta) \in \mathcal{P}(\mathbb{T})$ *then*

$$\mathrm{e}^{i(U + \arg a + X + Y)} \stackrel{\mathcal{D}}{=} \omega_H * \delta_a * \gamma_{\psi_b} * \pi_{\eta, g_{\mathbb{T}}},$$

where U, X *and* Y *are independent real random variables such that* U *is uniformly distributed on* $[0, 2\pi]$ *if* $H = \mathbb{T}$, U *is uniformly distributed on* $\{2\pi j/r : j = 0, 1, \ldots, r-1\}$ *if* $H = H_r$, X *has a normal distribution on* $\mathbb{R}$ *with zero mean and variance* b, *and the distribution of* Y *is the generalized Poisson measure* $\pi_{\arg\circ\eta,\, g_{\mathbb{R}}}$ *on* $\mathbb{R}$, *where the measure* $\arg\circ\eta$ *on* $\mathbb{R}$ *is defined by* $(\arg\circ\eta)(B) := \eta\big(\{x \in \mathbb{T} : \arg(x) \in B\}\big)$ *for all Borel subsets* B *of* $\mathbb{R}$.

Proof. Let U be a real random variable which is uniformly distributed on $[0, 2\pi]$. Then for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$, $\ell \neq 0$,

$$\mathrm{E}\,\chi_\ell(\mathrm{e}^{iU}) = \mathrm{E}\,\mathrm{e}^{i\ell U} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{i\ell x}\,\mathrm{d}x = 0.$$

Hence $\operatorname{E}\chi_\ell(\mathrm{e}^{iU}) = \widehat{\omega}_{\mathbb{T}}(\chi_\ell)$ for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$, and we obtain $\mathrm{e}^{iU} \stackrel{\mathcal{D}}{=} \omega_{\mathbb{T}}$.

Now let U be a real random variable which is uniformly distributed on $\{2\pi j/r : j = 0, 1, \dots, r-1\}$ with some $r \in \mathbb{N}$. Then for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$,

$$\operatorname{E}\chi_\ell(\mathrm{e}^{iU}) = \operatorname{E}\mathrm{e}^{i\ell U} = \frac{1}{r}\sum_{j=0}^{r-1} \mathrm{e}^{2\pi i\ell j/r} = \begin{cases} 1 & \text{if } r|\ell, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\operatorname{E}\chi_\ell(\mathrm{e}^{iU}) = \widehat{\omega}_{H_r}(\chi_\ell)$ for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$, and we obtain $\mathrm{e}^{iU} \stackrel{\mathcal{D}}{=} \omega_{H_r}$.

For $a \in \mathbb{T}$, we have $a = \mathrm{e}^{i\arg a}$, hence $\mathrm{e}^{i\arg a} \stackrel{\mathcal{D}}{=} \delta_a$.

For $b \in \mathbb{R}_+$, the Fourier transform of the Gauss measure γ_{ψ_b} has the form

$$\widehat{\gamma}_{\psi_b}(\chi_\ell) = \mathrm{e}^{-b\ell^2/2}, \qquad \chi_\ell \in \widehat{\mathbb{T}}, \quad \ell \in \mathbb{Z}.$$

For all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$,

$$\operatorname{E}\chi_\ell(\mathrm{e}^{iX}) = \operatorname{E}\mathrm{e}^{i\ell X} = \mathrm{e}^{-b\ell^2/2}.$$

Hence $\operatorname{E}\chi_\ell(\mathrm{e}^{iX}) = \gamma_{\psi_b}(\chi_\ell)$ for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$, and we obtain $\mathrm{e}^{iX} \stackrel{\mathcal{D}}{=} \gamma_{\psi_b}$.

For a Lévy measure $\eta \in \mathbb{L}(\mathbb{T})$, the Fourier transform of the generalized Poisson measure $\pi_{\eta, g_{\mathbb{T}}}$ has the form

$$\widehat{\pi}_{\eta, g_{\mathbb{T}}}(\chi_\ell) = \exp\left\{\int_{\mathbb{T}} \big(y^\ell - 1 - i\ell h(\arg y)\big) \mathrm{d}\eta(y)\right\}, \qquad \chi_\ell \in \widehat{\mathbb{T}}, \quad \ell \in \mathbb{Z}.$$

A measure $\widetilde{\eta}$ on $\mathbb{R}$ with values in $[0, +\infty]$ is a Lévy measure if and only if $\widetilde{\eta}(\{0\}) = 0$ and $\int_{\mathbb{R}} \min\{1, x^2\}\, \mathrm{d}\widetilde{\eta}(x) < \infty$. Consequently, $\arg\circ\eta$ is a Lévy measure on $\mathbb{R}$, and for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$,

$$\begin{aligned} \operatorname{E}\chi_\ell(\mathrm{e}^{iY}) = \operatorname{E}\mathrm{e}^{i\ell Y} &= \exp\left\{\int_{\mathbb{R}} \big(\mathrm{e}^{i\ell x} - 1 - i\ell h(x)\big)\, \mathrm{d}(\arg\circ\eta)(x)\right\} \\ &= \exp\left\{\int_{\mathbb{T}} \big(y^\ell - 1 - i\ell h(\arg y)\big)\, \mathrm{d}\eta(y)\right\}. \end{aligned}$$

Hence $\operatorname{E}\chi_\ell(\mathrm{e}^{iY}) = \widehat{\pi}_{\eta, g_{\mathbb{T}}}(\chi_\ell)$ for all $\chi_\ell \in \widehat{\mathbb{T}}$, $\ell \in \mathbb{Z}$, and we obtain $\mathrm{e}^{iY} \stackrel{\mathcal{D}}{=} \pi_{\eta, g_{\mathbb{T}}}$.

Finally, independence of U, X and Y implies

$$\begin{aligned} \operatorname{E}\chi(\mathrm{e}^{i(U+\arg a+X+Y)}) &= \operatorname{E}\chi(\mathrm{e}^{iU}) \cdot \chi(\mathrm{e}^{i\arg a}) \cdot \operatorname{E}\chi(\mathrm{e}^{iX}) \cdot \operatorname{E}\chi(\mathrm{e}^{iY}) \\ &= \widehat{\omega}_H(\chi)\, \widehat{\delta}_a(\chi)\, \widehat{\gamma}_{\psi_b}(\chi)\, \widehat{\pi}_{\eta, g_{\mathbb{T}}}(\chi) = (\omega_H * \delta_a * \gamma_{\psi_b} * \pi_{\eta, g_{\mathbb{T}}})\widehat{\ }(\chi) \end{aligned}$$

for all $\chi \in \widehat{\mathbb{T}}$, hence we obtain the statement. $\square$

4 Weakly infinitely divisible measures on the group of p–adic integers

Let p be a prime. The group of p–adic integers is

$$\Delta_p := \big\{(x_0, x_1, \dots) : x_j \in \{0, 1, \dots, p-1\} \text{ for all } j \in \mathbb{Z}_+\big\},$$

where the sum $z := x + y \in \Delta_p$ for $x, y \in \Delta_p$ is uniquely determined by the relationships

$$\sum_{j=0}^{d} z_j p^j \equiv \sum_{j=0}^{d} (x_j + y_j) p^j \qquad \bmod p^{d+1} \qquad \text{for all } d \in \mathbb{Z}_+.$$

For each $r \in \mathbb{Z}_+$, let

$$\Lambda_r := \{x \in \Delta_p : x_j = 0 \text{ for all } j \leqslant r - 1\}.$$

The family of sets $\{x + \Lambda_r : x \in \Delta_p,\ r \in \mathbb{Z}_+\}$ is an open subbasis for a topology on Δ_p under which Δ_p is a compact, totally disconnected Abelian T_0–topological group having a countable basis of its topology. For elementary facts about Δ_p we refer to the monographs Hewitt and Ross [8], Heyer [9] and Hofmann and Morris [10]. The character group of Δ_p is $\widehat{\Delta}_p = \{\chi_{d,\ell} : d \in \mathbb{Z}_+,\ \ell = 0, 1, \dots, p^{d+1} - 1\}$, where

$$\chi_{d,\ell}(x) := \mathrm{e}^{2\pi i \ell (x_0 + p x_1 + \cdots + p^d x_d)/p^{d+1}}, \quad x \in \Delta_p, \quad d \in \mathbb{Z}_+,\ \ell = 0, 1, \dots, p^{d+1} - 1.$$

The compact subgroups of Δ_p are Λ_r, $r \in \mathbb{Z}_+$ (see Hewitt and Ross [8, Example 10.16 (a)]).

A measure η on Δ_p with values in $[0, +\infty]$ is a Lévy measure if and only if $\eta(\{e\}) = 0$ and $\eta(\Delta_p \setminus \Lambda_r) < \infty$ for all $r \in \mathbb{Z}_+$.

Since the group Δ_p is totally disconnected, the only quadratic form on $\widehat{\Delta}_p$ is $\psi = 0$, and thus there is no nontrivial Gauss measure on Δ_p. Moreover, the function $g_{\Delta_p} : \Delta_p \times \widehat{\Delta}_p \to \mathbb{R}$, $g_{\Delta_p} = 0$ is a local inner product for Δ_p.

Now we prove that $\mathcal{I}(\Delta_p) \neq \mathcal{I}_{\mathrm{w}}(\Delta_p)$ by showing that there exists an element $x \in \Delta_p$ such that $\delta_x \notin \mathcal{I}(\Delta_p)$. Indeed, the Dirac measure at the element $(1, 0, \dots) \in \Delta_p$ is not infinitely divisible, since for each element $y \in \Delta_p$, the sum py has the form $(0, z_1, z_2, \dots) \in \Delta_p$ with some $z_i \in \{0, 1, \dots, p - 1\}$, $i \in \mathbb{N}$.

Our aim is to show that for a weakly infinitely divisible measure μ on Δ_p there exist integer–valued random variables $U_0, U_1, \dots$ and $Z_0, Z_1, \dots$ such that $U_0, U_1, \dots$ are independent of each other and of the sequence $Z_0, Z_1, \dots,$ moreover $U_0, U_1, \dots$ are uniformly distributed on a suitable subset of $\mathbb{Z}$, $(Z_0, \dots, Z_n)$ has a weakly infinitely divisible distribution on $\mathbb{Z}^{n+1}$ for all $n \in \mathbb{Z}_+$, and $\varphi(U_0 + Z_0,\ U_1 + Z_1, \dots) \stackrel{\mathcal{D}}{=} \mu$, where the mapping $\varphi : \mathbb{Z}^\infty \to \Delta_p$, uniquely defined by the relationships

$$\sum_{j=0}^{d} y_j p^j \equiv \sum_{j=0}^{d} \varphi(y)_j p^j \qquad \bmod p^{d+1} \qquad \text{for all } d \in \mathbb{Z}_+, \tag{4.1}$$

is a continuous homomorphism from the Abelian topological group $\mathbb{Z}^\infty$ (furnished with the product topology) onto Δ_p. (Note that $\mathbb{Z}^\infty$ is not locally compact.) Continuity of φ follows from the definition of the product topology and the fact that

$$\varphi^{-1}(x + \Lambda_r) = \{y \in \mathbb{Z}^\infty : (y_0, y_1, \dots, y_{r-1}) \in F_{x,r}\}$$

for all $x \in \Delta_p$, $r \in \mathbb{Z}_+$, where $F_{x,r}$ is a suitable (open) subset of $\mathbb{Z}^r$.

4.1 Theorem. *If* $(\Lambda_r, a, 0, \eta) \in \mathcal{P}(\Delta_p)$ *then*

$$\varphi(U_0 + a_0 + Y_0,\, U_1 + a_1 + Y_1, \dots) \stackrel{\mathcal{D}}{=} \omega_{\Lambda_r} * \delta_a * \pi_{\eta,\, g_{\Delta_p}},$$

where $U_0, U_1, \dots$ *and* $Y_0, Y_1, \dots$ *are integer–valued random variables such that* $U_0, U_1, \dots$ *are independent of each other and of the sequence* $Y_0, Y_1, \dots,$ *moreover* $U_0 = \dots = U_{r-1} = 0$ *and* $U_r, U_{r+1}, \dots$ *are uniformly distributed on* $\{0, 1, \dots, p-1\}$, *and the distribution of* $(Y_0, \dots, Y_n)$ *is the compound Poisson measure* $\mathrm{e}(\eta_{n+1})$ *for all* $n \in \mathbb{Z}_+$, *where the measure* η_{n+1} *on* $\mathbb{Z}^{n+1}$ *is defined by* $\eta_{n+1}(\{0\}) := 0$ *and* $\eta_{n+1}(\ell) := \eta(\{x \in \Delta_p : (x_0, x_1, \dots, x_n) = \ell\})$ *for all* $\ell \in \mathbb{Z}^{n+1} \setminus \{0\}$.

Proof. Since $U_0, U_1, \dots$ and $Y_0, Y_1, \dots$ are integer–valued random variables and the mapping $\varphi : \mathbb{Z}^\infty \to \Delta_p$ is continuous, we obtain that $\varphi(U_0 + a_0 + Y_0,\, U_1 + a_1 + Y_1, \dots)$ is a random element with values in Δ_p.

First we show $\varphi(U) \stackrel{\mathcal{D}}{=} \omega_{\Lambda_r}$, where $U := (U_0, U_1, \dots)$. By (4.1) we obtain

$$\mathrm{E}\,\chi_{d,\ell}(\varphi(U)) = \mathrm{E}\,\mathrm{e}^{2\pi i\ell(\varphi(U)_0 + p\varphi(U)_1 + \dots + p^d\varphi(U)_d)/p^{d+1}} = \mathrm{E}\,\mathrm{e}^{2\pi i\ell(U_0 + pU_1 + \dots + p^dU_d)/p^{d+1}}$$

$$= \begin{cases} \dfrac{1}{p^{d-r+1}} \displaystyle\sum_{j_r=0}^{p-1} \dots \sum_{j_d=0}^{p-1} \mathrm{e}^{2\pi i\ell(p^r j_r + \dots + p^d j_d)/p^{d+1}} = 0 & \text{if } d \geqslant r \text{ and } p^{d+1-r} \nmid \ell, \\ 1 & \text{otherwise} \end{cases}$$

for all $d \in \mathbb{Z}_+$ and $\ell = 0, 1, \dots, p^{d+1} - 1$. Hence $\mathrm{E}\,\chi_{d,\ell}(\varphi(U)) = \widehat{\omega}_{\Lambda_r}(\chi_{d,\ell})$ for all $d \in \mathbb{Z}_+$ and $\ell = 0, 1, \dots, p^{d+1} - 1$, and we obtain $\varphi(U) \stackrel{\mathcal{D}}{=} \omega_{\Lambda_r}$.

For $a \in \Delta_p$, we have $a = \varphi(a_0, a_1, \dots)$, hence $\varphi(a_0, a_1, \dots) \stackrel{\mathcal{D}}{=} \delta_a$.

For a Lévy measure $\eta \in \mathbb{L}(\Delta_p)$, the Fourier transform of the generalized Poisson measure $\pi_{\eta,\, g_{\Delta_p}}$ has the form

$$\widehat{\pi}_{\eta,\, g_{\Delta_p}}(\chi_{d,\ell}) = \exp\left\{ \int_{\Delta_p} \left(\mathrm{e}^{2\pi i\ell(x_0 + px_1 + \dots + p^d x_d)/p^{d+1}} - 1\right) \mathrm{d}\eta(x) \right\}$$

for all $d \in \mathbb{Z}_+$ and $\ell = 0, 1, \dots, p^{d+1} - 1$. We have $\eta_{n+1}(\mathbb{Z}^{n+1}) = \eta(\Delta_p \setminus \Lambda_{n+1}) < \infty$, hence η_{n+1} is a bounded measure on $\mathbb{Z}^{n+1}$, and the compound Poisson measure $\mathrm{e}(\eta_{n+1})$ on $\mathbb{Z}^{n+1}$ is defined. The character group of $\mathbb{Z}^{n+1}$ is $(\mathbb{Z}^{n+1})\widehat{\ } = \{\chi_{z_0, z_1, \dots, z_n} : z_0, z_1, \dots, z_n \in \mathbb{T}\}$, where $\chi_{z_0, z_1, \dots, z_n}(\ell_0, \ell_1, \dots, \ell_n) := z_0^{\ell_0} z_1^{\ell_1} \cdots z_n^{\ell_n}$ for all $(\ell_0, \ell_1, \dots, \ell_n) \in \mathbb{Z}^{n+1}$. The family of measures $\{\mathrm{e}(\eta_{n+1}) : n \in \mathbb{Z}_+\}$ is compatible, since $\mathrm{e}(\eta_{n+2})(\{\ell\} \times \mathbb{Z}) = \mathrm{e}(\eta_{n+1})(\{\ell\})$ for all $\ell \in \mathbb{Z}^{n+1}$ and $n \in \mathbb{Z}_+$. Indeed, this is a consequence of

$$(\mathrm{e}(\eta_{n+2}))\widehat{\ }(\chi_{z_0, z_1, \dots, z_n, 1}) = (\mathrm{e}(\eta_{n+1}))\widehat{\ }(\chi_{z_0, z_1, \dots, z_n})$$

for all $z_0, z_1, \dots, z_n \in \mathbb{T}$, which follows from

$$\int_{\mathbb{Z}^{n+2}} (z_0^{\ell_0} z_1^{\ell_1} \cdots z_n^{\ell_n} - 1)\, \mathrm{d}\eta_{n+2}(\ell_0, \ell_1, \dots, \ell_n, \ell_{n+1})$$

$$= \int_{\mathbb{Z}^{n+1}} (z_0^{\ell_0} z_1^{\ell_1} \cdots z_n^{\ell_n} - 1)\, \mathrm{d}\eta_{n+1}(\ell_0, \ell_1, \dots, \ell_n)$$

for all $z_0, z_1, \ldots, z_n \in \mathbb{T}$, where both sides are equal to $\int_{\Delta_p} (z_0^{x_0} z_1^{x_1} \cdots z_n^{x_n} - 1) \, \mathrm{d}\eta(x)$. This integral is finite, since

$$\begin{aligned}\int_{\Delta_p} |z_0^{x_0} z_1^{x_1} \cdots z_n^{x_n} - 1| \, \mathrm{d}\eta(x) &= \int_{\Delta_p \setminus \Lambda_{n+1}} |z_0^{x_0} z_1^{x_1} \cdots z_n^{x_n} - 1| \, \mathrm{d}\eta(x) \\ &\leqslant 2\eta(\Delta_p \setminus \Lambda_{n+1}) < \infty.\end{aligned}$$

By Kolmogorov's Consistency Theorem, there exists a sequence $Y_0, Y_1, \ldots$ of integer–valued random variables such that the distribution of $(Y_0, \ldots, Y_n)$ is the compound Poisson measure $\mathrm{e}(\eta_{n+1})$ for all $n \in \mathbb{Z}_+$. For all $d \in \mathbb{Z}_+$ and $\ell = 0, 1, \ldots, p^{d+1} - 1$ we have

$$\begin{aligned}\mathrm{E}\, \chi_{d,\ell}(\varphi(Y_0, Y_1, \ldots)) &= \mathrm{E}\, \mathrm{e}^{2\pi i \ell (Y_0 + pY_1 + \cdots + p^d Y_d)/p^{d+1}} \\ &= \exp\left\{ \int_{\mathbb{Z}^{d+1}} \left(\mathrm{e}^{2\pi i \ell (\ell_0 + p\ell_1 + \cdots + p^d \ell_d)/p^{d+1}} - 1 \right) \mathrm{d}\eta_{d+1}(\ell_0, \ell_1, \ldots, \ell_d) \right\} \\ &= \exp\left\{ \int_{\Delta_p} \left(\mathrm{e}^{2\pi i \ell (x_0 + px_1 + \cdots + p^d x_d)/p^{d+1}} - 1 \right) \mathrm{d}\eta(x) \right\}.\end{aligned}$$

Hence $\mathrm{E}\, \chi_{d,\ell}(\varphi(Y_0, Y_1, \ldots)) = \widehat{\pi}_{\eta, g_{\Delta_p}}(\chi_{d,\ell})$ for all $d \in \mathbb{Z}_+$ and $\ell = 0, 1, \ldots, p^{d+1} - 1$, and we obtain $\varphi(Y_0, Y_1, \ldots) \stackrel{\mathcal{D}}{=} \pi_{\eta, g_{\Delta_p}}$.

Since the sequences $U_0, U_1, \ldots$ and $Y_0, Y_1, \ldots$ are independent and the mapping $\varphi : \mathbb{Z}^\infty \to \Delta_p$ is a homomorphism, we have

$$\begin{aligned}&\mathrm{E}\, \chi(\varphi(U_0 + a_0 + Y_0,\, U_1 + a_1 + Y_1, \ldots)) \\ &= \mathrm{E}\, \chi(\varphi(U_0, U_1, \ldots)) \cdot \chi(\varphi(a_0, a_1, \ldots)) \cdot \mathrm{E}\, \chi(\varphi(Y_0, Y_1, \ldots)) \\ &= \widehat{\omega}_{\Lambda_r}(\chi)\, \widehat{\delta}_a(\chi)\, \widehat{\pi}_{\eta, g_{\Delta_p}}(\chi) = (\omega_{\Lambda_r} * \delta_a * \pi_{\eta, g_{\Delta_p}})\widehat{\ }(\chi)\end{aligned}$$

for all $\chi \in \widehat{\Delta}_p$, and we obtain the statement. □

5 Weakly infinitely divisible measures on the p–adic solenoid

Let p be a prime. The p–adic solenoid is a subgroup of $\mathbb{T}^\infty$, namely,

$$S_p = \left\{ (y_0, y_1, \ldots) \in \mathbb{T}^\infty : y_j = y_{j+1}^p \text{ for all } j \in \mathbb{Z}_+ \right\}.$$

This is a compact Abelian T_0–topological group having a countable basis of its topology. For elementary facts about S_p we refer to the monographs Hewitt and Ross [8], Heyer [9] and Hofmann and Morris [10]. The character group S_p is $\widehat{S}_p = \{\chi_{d,\ell} : d \in \mathbb{Z}_+, \ell \in \mathbb{Z}\}$, where

$$\chi_{d,\ell}(y) := y_d^\ell, \qquad y \in S_p, \quad d \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z}.$$

The set of all quadratic forms on $\widehat{S}_p$ is $\mathrm{q}_+\big(\widehat{S}_p\big) = \{\psi_b : b \in \mathbb{R}_+\}$, where

$$\psi_b(\chi_{d,\ell}) := \frac{b\ell^2}{p^{2d}}, \qquad d \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z}, \quad b \in \mathbb{R}_+.$$

A measure η on S_p with values in $[0, +\infty]$ is a Lévy measure if and only if $\eta(\{e\}) = 0$ and $\int_{S_p} (\arg y_0)^2 \,\mathrm{d}\eta(y) < \infty$. The function $g_{S_p} : S_p \times \widehat{S}_p \to \mathbb{R}$,

$$g_{S_p}(y, \chi_{d,\ell}) := \frac{\ell h(\arg y_0)}{p^d}, \qquad y \in S_p, \quad d \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z},$$

is a local inner product for S_p.

Using a result of Carnal [5], Becker-Kern [3] showed that all the Dirac measures on S_p are infinitely divisible, which implies $\mathcal{I}(S_p) = \mathcal{I}_{\mathrm{w}}(S_p)$.

Our aim is to show that for a weakly infinitely divisible measure μ on S_p without an idempotent factor there exist real random variables $Z_0, Z_1, \ldots$ such that $(Z_0, \ldots, Z_n)$ has a weakly infinitely divisible distribution on $\mathbb{R} \times \mathbb{Z}^n$ for all $n \in \mathbb{Z}_+$, and $\varphi(Z_0,\, Z_1, \ldots) \stackrel{\mathcal{D}}{=} \mu$, where the mapping $\varphi : \mathbb{R} \times \mathbb{Z}^\infty \to S_p$, defined by

$$\begin{aligned} &\varphi(y_0, y_1, y_2, \ldots) \\ &\quad := \big(\mathrm{e}^{iy_0},\, \mathrm{e}^{i(y_0 + 2\pi y_1)/p},\, \mathrm{e}^{i(y_0 + 2\pi y_1 + 2\pi y_2 p)/p^2},\, \mathrm{e}^{i(y_0 + 2\pi y_1 + 2\pi y_2 p + 2\pi y_3 p^2)/p^3}, \ldots\big) \end{aligned}$$

for $(y_0, y_1, y_2, \ldots) \in \mathbb{R} \times \mathbb{Z}^\infty$, is a continuous homomorphism from the Abelian topological group $\mathbb{R} \times \mathbb{Z}^\infty$ (furnished with the product topology) onto S_p. Continuity of φ follows from the fact that S_p as a subspace of $\mathbb{T}^\infty$ is furnished with the relative topology. Note that $\mathbb{R} \times \mathbb{Z}^\infty$ is not locally compact, but $\mathbb{R} \times \mathbb{Z}^n$ is a locally compact Abelian T_0–topological group having a countable basis of its topology for all $n \in \mathbb{Z}_+$. The character group of $\mathbb{R} \times \mathbb{Z}^n$ is $(\mathbb{R} \times \mathbb{Z}^n)^\wedge = \{\chi_{y,z} : y \in \mathbb{R},\, z \in \mathbb{T}^n\}$, where $\chi_{y,z}(x, \ell) := \mathrm{e}^{iyx} z_1^{\ell_1} \cdots z_n^{\ell_n}$ for all $x, y \in \mathbb{R}$, $z = (z_1, \ldots, z_n) \in \mathbb{T}^n$ and $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n$. The function $g_{\mathbb{R} \times \mathbb{Z}^n}\big((x, \ell), \chi_{y,z}\big) := yh(x)$ is a local inner product for $\mathbb{R} \times \mathbb{Z}^n$.

We also find independent real random variables $U_0, U_1, \ldots$ such that $U_0, U_1, \ldots$ are uniformly distributed on suitable subsets of $\mathbb{R}$ and $\varphi(U_0,\, U_1, \ldots) \stackrel{\mathcal{D}}{=} \omega_{S_p}$.

5.1 Theorem. *If* $(\{e\}, a, \psi_b, \eta) \in \mathcal{P}(S_p)$ *then*

$$\begin{aligned} &\varphi(\tau(a)_0 + X_0 + Y_0,\, \tau(a)_1 + Y_1,\, \tau(a)_2 + Y_2, \ldots) \\ &= \Big(a_0 \mathrm{e}^{i(X_0 + Y_0)},\, a_1 \mathrm{e}^{i(X_0 + Y_0 + 2\pi Y_1)/p},\, a_2 \mathrm{e}^{i(X_0 + Y_0 + 2\pi Y_1 + 2\pi Y_2 p)/p^2}, \ldots\Big) \\ &\stackrel{\mathcal{D}}{=} \delta_a * \gamma_{\psi_b} * \pi_{\eta, g_{S_p}}, \end{aligned}$$

where the mapping $\tau : S_p \to \mathbb{R} \times \mathbb{Z}^\infty$ *is defined by*

$$\tau(x) := \left(\arg x_0,\, \frac{p \arg x_1 - \arg x_0}{2\pi},\, \frac{p \arg x_2 - \arg x_1}{2\pi}, \ldots\right)$$

for $x = (x_0, x_1, \ldots) \in S_p$. *Here* X_0, Y_0 *are real random variables and* $Y_1, Y_2, \ldots$ *are integer–valued random variables such that* X_0 *is independent of the sequence* $Y_0, Y_1, \ldots,$ *the* *variable* X_0 *has a normal distribution with zero mean and variance* b, *and the distribution of* $(Y_0, \dots, Y_n)$ *is the generalized Poisson measure* $\pi_{\eta_{n+1}, g_{\mathbb{R}\times\mathbb{Z}^n}}$ *for all* $n \in \mathbb{Z}_+$, *where the measure* η_{n+1} *on* $\mathbb{R}\times\mathbb{Z}^n$ *is defined by* $\eta_{n+1}(\{0\}) := 0$ *and*

$$\eta_{n+1}(B \times \{\ell\}) := \eta\big(\{x \in S_p : \tau(x)_0 \in B,\, (\tau(x)_1, \dots, \tau(x)_n) = \ell\}\big)$$

for all Borel subsets B *of* $\mathbb{R}$ *and for all* $\ell \in \mathbb{Z}^n$ *with* $0 \notin B \times \{\ell\}$.

Moreover,

$$\varphi(U_0, U_1, \dots) \stackrel{\mathcal{D}}{=} \omega_{S_p},$$

where $U_0, U_1, \dots$ *are independent real random variables such that* U_0 *is uniformly distributed on* $[0, 2\pi]$ *and* $U_1, U_2, \dots$ *are uniformly distributed on* $\{0, 1, \dots, p-1\}$.

Proof. Since X_0, Y_0 and $U_0, U_1, \dots$ are real random variables and $Y_1, Y_2, \dots$ are integer–valued random variables and the mapping $\varphi : \mathbb{R}\times\mathbb{Z}^\infty \to S_p$ is continuous, we obtain that $\varphi(\tau(a)_0 + X_0 + Y_0,\, \tau(a)_1 + Y_1,\, \tau(a)_2 + Y_2, \dots)$ and $\varphi(U_0, U_1, \dots)$ are random elements with values in S_p.

For $a \in S_p$, we have $a = \varphi(\tau(a))$, hence $\varphi(\tau(a)) \stackrel{\mathcal{D}}{=} \delta_a$.

For $b \in \mathbb{R}_+$, the Fourier transform of the Gauss measure γ_{ψ_b} has the form

$$\widehat{\gamma}_{\psi_b}(\chi_{d,\ell}) = \exp\left\{-\frac{b\ell^2}{2p^{2d}}\right\}, \qquad d \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z}.$$

For all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$,

$$\mathrm{E}\,\chi_{d,\ell}(\varphi(X_0, 0, 0, \dots)) = \mathrm{E}\,\mathrm{e}^{i\ell X_0/p^d} = \exp\left\{-\frac{b\ell^2}{2p^{2d}}\right\}.$$

Hence $\mathrm{E}\,\chi_{d,\ell}(\varphi(X_0, 0, 0, \dots)) = \widehat{\gamma}_{\psi_b}(\chi_{d,\ell})$ for all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$, and we obtain that $\varphi(X_0, 0, 0, \dots) \stackrel{\mathcal{D}}{=} \gamma_{\psi_b}$.

For a Lévy measure $\eta \in \mathbb{L}(S_p)$, the Fourier transform of the generalized Poisson measure $\pi_{\eta, g_{S_p}}$ has the form

$$\widehat{\pi}_{\eta, g_{S_p}}(\chi_{d,\ell}) = \exp\left\{\int_{S_p} \big(y_d^\ell - 1 - i\ell h(\arg y_0)/p^d\big)\, \mathrm{d}\eta(y)\right\}$$

for all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$. A measure $\widetilde{\eta}$ on $\mathbb{R}\times\mathbb{Z}^n$ with values in $[0, +\infty]$ is a Lévy measure if and only if $\widetilde{\eta}(\{0\}) = 0$, $\widetilde{\eta}(\{(x,\ell) \in \mathbb{R}\times\mathbb{Z}^n : |x| \geqslant \varepsilon \text{ or } \ell \neq 0\}) < \infty$ for all $\varepsilon > 0$, and $\int_{\mathbb{R}\times\mathbb{Z}^n} h(x)^2\, \mathrm{d}\widetilde{\eta}(x,\ell) < \infty$. We have

$$\begin{aligned}
&\eta_{n+1}(\{(x,\ell) \in \mathbb{R}\times\mathbb{Z}^n : |x| \geqslant \varepsilon \text{ or } \ell \neq 0\}) \\
&= \eta(\{y \in S_p : |\arg y_0| \geqslant \varepsilon \text{ or } (\tau(y)_1, \dots, \tau(y)_n) \neq 0\}) = \eta(S_p \setminus N_{\varepsilon,n}) < \infty
\end{aligned}$$

for all $\varepsilon \in (0, \pi)$, where

$$N_{\varepsilon,n} := \{y \in S_p : |\arg y_0| < \varepsilon,\, |\arg y_1| < \varepsilon/p, \dots, |\arg y_n| < \varepsilon/p^n\}.$$

Moreover, $\int_{\mathbb{R}\times\mathbb{Z}^n} h(x)^2 \,\mathrm{d}\eta_{n+1}(x,\ell) = \int_{S_p} h(\arg y_0)^2 \,\mathrm{d}\eta(y) < \infty$, since η is a Lévy measure on S_p. Consequently, η_{n+1} is a Lévy measure on $\mathbb{R}\times\mathbb{Z}^n$. The family of measures $\{\pi_{\eta_{n+1},\, g_{\mathbb{R}\times\mathbb{Z}^n}} : n \in \mathbb{Z}_+\}$ is compatible, since $\pi_{\eta_{n+2},\, g_{\mathbb{R}\times\mathbb{Z}^{n+1}}}(\{x\}\times\mathbb{Z}) = \pi_{\eta_{n+1},\, g_{\mathbb{R}\times\mathbb{Z}^n}}(\{x\})$ for all $x \in \mathbb{R}\times\mathbb{Z}^{n+1}$ and $n \in \mathbb{Z}_+$. Indeed, this is a consequence of

$$(\pi_{\eta_{n+2},\, g_{\mathbb{R}\times\mathbb{Z}^{n+1}}})\widehat{\ }(\chi_{y,z_1,\dots,z_n,1}) = (\pi_{\eta_{n+1},\, g_{\mathbb{R}\times\mathbb{Z}^n}})\widehat{\ }(\chi_{y,z_1,\dots,z_n})$$

for all $y \in \mathbb{R}$, $z_1,\dots,z_n \in \mathbb{T}$, which follows from

$$\int_{\mathbb{R}\times\mathbb{Z}^{n+1}} \big(\mathrm{e}^{iyx} z_1^{\ell_1}\cdots z_n^{\ell_n} - 1 - iyh(x)\big)\,\mathrm{d}\eta_{n+2}(x,\ell_1,\dots,\ell_n,\ell_{n+1})$$

$$= \int_{\mathbb{R}\times\mathbb{Z}^n} \big(\mathrm{e}^{iyx} z_1^{\ell_1}\cdots z_n^{\ell_n} - 1 - iyh(x)\big)\,\mathrm{d}\eta_{n+1}(x,\ell_1,\dots,\ell_n)$$

for all $y \in \mathbb{R}$, $z_1,\dots,z_n \in \mathbb{T}$, where both sides are equal to

$$I := \int_{S_p} \big(\mathrm{e}^{iy\arg x_0} z_1^{(p\arg x_1 - \arg x_0)/(2\pi)} \cdots z_n^{(p\arg x_n - \arg x_{n-1})/(2\pi)}$$
$$- 1 - iyh(\arg x_0)\big)\,\mathrm{d}\eta(x).$$

This integral is finite. Indeed, for all $x \in N_{\varepsilon,n}$ and $0<\varepsilon<\pi/2$ we have $p\arg x_k = \arg x_{k-1}$ for each $k = 1,\dots,n$, hence

$$|I| \leqslant (2+\pi|y|)\,\eta(S_p\setminus N_{\varepsilon,n}) + \int_{N_{\varepsilon,n}} |\mathrm{e}^{iy\arg x_0} - 1 - iy\arg x_0|\,\mathrm{d}\eta(x)$$

$$\leqslant (2+\pi|y|)\,\eta(S_p\setminus N_{\varepsilon,n}) + \frac{1}{2}\int_{N_{\varepsilon,n}} (\arg x_0)^2\,\mathrm{d}\eta(x) < \infty,$$

since η is a Lévy measure on S_p. By Kolmogorov's Consistency Theorem, there exist a real random variable Y_0 and a sequence $Y_1, Y_2, \dots$ of integer–valued random variables such that the distribution of $(Y_0,\dots,Y_n)$ is the generalized Poisson measure $\pi_{\eta_{n+1},\, g_{\mathbb{R}\times\mathbb{Z}^n}}$ for all $n \in \mathbb{Z}_+$. For all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$, we have

$$\operatorname{E}\chi_{d,\ell}(\varphi(Y_0,Y_1,\dots)) = \operatorname{E}\mathrm{e}^{i\ell(Y_0+2\pi Y_1+\dots+2\pi Y_d p^{d-1})/p^d}$$

$$= \exp\left\{\int_{\mathbb{R}\times\mathbb{Z}^d} \big(\mathrm{e}^{i\ell(x+2\pi\ell_1+\dots+2\pi\ell_d p^{d-1})/p^d} - 1 - i\ell h(x)/p^d\big)\,\mathrm{d}\eta_{d+1}(x,\ell_1,\dots,\ell_d)\right\}$$

$$= \exp\left\{\int_{S_p} \big(y_d^\ell - 1 - i\ell h(\arg y_0)/p^d\big)\,\mathrm{d}\eta(y)\right\}.$$

Hence $\operatorname{E}\chi_{d,\ell}(\varphi(Y_0,Y_1,\dots)) = \widehat{\pi}_{\eta,\, g_{S_p}}(\chi_{d,\ell})$ for all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$, and we obtain $\varphi(Y_0,Y_1,\dots) \overset{\mathcal{D}}{=} \pi_{\eta,\, g_{S_p}}$.

Since the sequence $Y_0, Y_1, \dots$ and the random variable X_0 are independent and the mapping $\varphi : \mathbb{R}\times\mathbb{Z}^\infty \to S_p$ is a homomorphism, we get

$$\operatorname{E}\chi(\varphi(\tau(a)_0 + X_0 + Y_0,\ \tau(a)_1 + Y_1,\ \tau(a)_2 + Y_2,\dots))$$
$$= \chi(\varphi(\tau(a)_0,\tau(a)_1,\dots))\cdot \operatorname{E}\chi(\varphi(X_0,0,0,\dots))\cdot \operatorname{E}\chi(\varphi(Y_0,Y_1,\dots))$$
$$= \widehat{\delta}_a(\chi)\,\widehat{\gamma}_{\psi_b}(\chi)\,\widehat{\pi}_{\eta,\, g_{S_p}}(\chi) = (\delta_a * \gamma_{\psi_b} * \pi_{\eta,\, g_{S_p}})\widehat{\ }(\chi)$$

for all $\chi \in \widehat{S}_p$, and we obtain the first statement.

For all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z} \setminus \{0\}$,

$$\begin{aligned}
\operatorname{E} \chi_{d,\ell}(\varphi(U_0, U_1, \dots)) &= \operatorname{E} \mathrm{e}^{i\ell(U_0 + 2\pi U_1 + \dots + 2\pi U_d p^{d-1})/p^d} \\
&= \frac{1}{2\pi p^d} \int_0^{2\pi} \mathrm{e}^{i\ell x/p^d} \, \mathrm{d}x \sum_{j_0=0}^{p-1} \dots \sum_{j_{d-1}=0}^{p-1} \mathrm{e}^{2\pi i\ell(j_0 + j_1 p + \dots + j_{d-1} p^{d-1})/p^d} = 0.
\end{aligned}$$

Hence $\operatorname{E} \chi_{d,\ell}(\varphi(U_0, U_1, \dots)) = \widehat{\omega}_{S_p}(\chi_{d,\ell})$ for all $d \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$, and we obtain $\varphi(U_0, U_1, \dots) \stackrel{\mathcal{D}}{=} \omega_{S_p}$. □

Acknowledgments. The authors have been supported by the Hungarian Scientific Research Fund under Grant No. OTKA–T048544/2005. The first author has been also supported by the Hungarian Scientific Research Fund under Grant No. OTKA–F046061/2004.

References

[1] M. Barczy, Some questions of probability theory on special topological groups. *Ph.D. Thesis*, University of Debrecen, Hungary (2006), http://www.inf.unideb.hu/valseg/dolgozok/barczy/phdthesis_barczy.pdf

[2] M. Barczy, A. Bendikov and G. Pap, Limit theorems on locally compact Abelian groups, to appear in *Mathematische Nachrichten.* See also, arXiv: math/0702078v1 (2007).

[3] P. Becker-Kern, Explicit representation of roots on p–adic solenoids and non-uniqueness of embeddability into rational one-parameter subgroups, *Proceedings of the Indian Academy of Sciences (Mathematical Sciences)* **117**(4) (2007), 443-455.

[4] M. S. Bingham, Central limit theory on locally compact abelian groups, *In: Probability measures on groups and related structures, XI. Proceedings Oberwolfach, 1994, pp. 14–37*, (World Sci. Publishing, NJ) (1995).

[5] H. Carnal, Unendlich oft teilbare Wahrscheinlichkeitsverteilungen auf kompakten Gruppen, *Mathematische Annalen* **153** (1964), 351-383.

[6] D. V. Chistyakov, Fractal geometry for images of continuous map of p-adic numbers and p-adic solenoids into Euclidean spaces, *arXiv: math/0202089v1* (2002).

[7] J. Gaiser, Konvergenz stochastischer prozesse mit werten in einer lokalkompakten Abelschen gruppe. *Ph.D. Thesis*, Universität Tübingen, Germany (1994).

[8] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis (Berlin, Göttingen, Heidelberg: Springer–Verlag) (1963).

[9] H. Heyer, Probability Measures on Locally Compact Groups (Berlin, Heidelberg, New York: Springer–Verlag) (1977).

[10] K. H. Hofmann and S. A. Morris, The sructure of compact groups, 2nd Revised edition (Walter de Gruyter) (2006).

[11] K. R. Parthasarathy, Probability measures on metric spaces (New York and London: Academic Press) (1967).

[12] K. Telöken, Grenzwertsätze für wahrscheinlichkeitsmasse auf total unzusammenhängenden gruppen. *Ph.D. Thesis*, Universität Dortmund, Germany (1995).

[13] J. G. Wendel, Haar measure and the semigroup of measures on a compact group, *Proc. Amer. Math. Soc.* **5** (1954), 923–929.

[14] K. Yasuda, On infinitely divisible distributions on locally compact Abelian groups, *Journal of Theoretical Probability* **13**(3) (2000), 635-657.