ON THE USE OF MAJORITY FOR INVESTIGATING PRIMENESS OF 3-PERMUTABILITY

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ABSTRACT. We have recently published a result that $\boldsymbol{n}$-permutability is not join-prime in the lattice of interpretability types of varieties whenever $\boldsymbol{n} \geq 5$. In the proof we showed that if $\boldsymbol{n} \geq 5$, then the join of a properly chosen finitely generated non-$\boldsymbol{n}$-permutable variety and the variety $\mathscr{M}$ defined by the majority identities is $\boldsymbol{n}$-permutable. In the present note, we prove that the join of any locally finite non-3-permutable variety with $\mathscr{M}$ is non-3-permutable. We also prove that the join of any non-2-permutable variety with $\mathscr{M}$ is non-2-permutable. Our non-3-permutable result gives that one has to use a non-locally finite non-3-permutable variety $\mathscr{L}$ if they want to prove that 3-permutability is not join-prime by arguing that $\mathscr{L} \vee \mathscr{M}$ is 3-permutable.

1. INTRODUCTION

In this note we investigate the 3-permutability property in the lattice of interpretability types of varieties. Before launching into details, we present some of the basic definitions and concepts of the theory.

Let Γ be a set of identities over a certain signature of a variety. We say that Γ *interprets in a variety* $\mathscr{K}$ if by replacing every occurrence of each operation symbol s in the identities of Γ by a term t_s of $\mathscr{K}$ such that s and t_s have the same arity, the so obtained set of identities holds in $\mathscr{K}$. A *variety* $\mathscr{K}_1$ *interprets in a variety* $\mathscr{K}_2$ if there is a set of identities Γ that defines $\mathscr{K}_1$ and interprets in $\mathscr{K}_2$.

As easily seen, interpretability is a quasiorder on the class of varieties. The blocks of interpretability are called the *interpretability types*. In [2], Garcia and Taylor introduced the *lattice of interpretability types of varieties* that is obtained by taking the quotient of the class of varieties quasiordered by interpretability and its related equivalence relation. The join in this lattice is described as follows. Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be two varieties of disjoint signatures. Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be defined by the sets Σ_1 and Σ_2 of identities, respectively. Their *join* $\mathscr{K}_1 \vee \mathscr{K}_2$ is the variety defined by $\Sigma_1 \cup \Sigma_2$. The join, so defined, is compatible with the interpretability relation of varieties, and naturally yields the definition of the join operation in the lattice of interpretability types of varieties.

Let $\boldsymbol{n} \geq 2$ be an integer. An *algebra* $\mathbf{A}$ *is congruence* $\boldsymbol{n}$*-permutable*, if for any two congruences α and β of $\mathbf{A}$, $\alpha\beta\cdots = \beta\alpha\dots$ where each side of the equality consists of $\boldsymbol{n}$ alternating factors of α and β. A *variety is* $\boldsymbol{n}$*-permutable* if all of its members are congruence $\boldsymbol{n}$-permutable. An *k-ary operation* f *is idempotent* if it satisfies the identity $f(x,x,\dots,x) = x$. A *variety is idempotent* if it satisfies the preceding identity for every operation symbol f. An *identity is linear* if each of the two terms that determine the identity is a variable or a term that contains a single

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occurrence of a single operation symbol. A *variety is linear* if it is defined by linear identities.

In [2] Garcia and Taylor formulated the conjecture that 2-permutability is join-prime in the lattice of interpretability types of varieties. In [10] Tschantz announced a proof of the conjecture. However, his proof has remained unpublished.

Here we list some results of positive flavor related to Garcia and Taylor's conjecture. In Lemma 2.8 of [7], Kearnes and Tschantz basically gave a proof that 2-permutability is join-prime in the lattice of interpretability types of idempotent varieties. In [9], Opršal proved for any $\boldsymbol{n}$ that $\boldsymbol{n}$-permutability is join-prime in the lattice of interpretability types of linear varieties. In [11] Valeriote and Willard, verified that the interpretability types of the idempotent $\boldsymbol{n}$-permutable varieties for $\boldsymbol{n} \geq 2$ form a prime filter in the lattice of interpretability types of idempotent varieties. A similar result was proved by Opršal in [9] for linear varieties. In [1], Chicco proved that the join of any two locally finite idempotent non-3-permutable varieties is non-3-permutable in the lattice of interpretability types of varieties.

In [4] we gave some results of negative flavor related to Taylor's conjecture in the general case (where idempotency and linearity are not assumed). We proved that the filter of the interpretability types of the $\boldsymbol{n}$-permutable varieties where $\boldsymbol{n}$ runs through the integers greater than one is not prime in the lattice of interpretability types of varieties. We also proved that for any $\boldsymbol{n} \geq 5$, $\boldsymbol{n}$-permutability is not join-prime in the lattice of interpretability types of varieties. The questions whether 3- and 4-permutability are prime remained open. Our main result in the present note is related to the question on the primeness of 3-permutability. To give some motivation for our new approach, we first delineate the main proof in [4].

A k-ary operation f, $k \geq 3$, is a *near-unanimity operation* if it satisfies the identities

$$f(y,x,\dots,x) = f(x,y,\dots,x) = \cdots = f(x,x,\dots,y) = x.$$

A ternary near-unanimity operation is called a *majority operation*. We call the identities in the definition of a majority operation *majority identities*. In [4], for a ternary operation symbol m, we let $\mathscr{M}$ be the variety defined by the set of majority identities for m. The variety $\mathscr{M}$ is well known to be not $\boldsymbol{n}$-permutable for any $\boldsymbol{n}$. Then we presented a finitely generated variety $\mathscr{L}$ that is also not $\boldsymbol{n}$-permutable for any $\boldsymbol{n}$, and we proved that that $\mathscr{L} \vee \mathscr{M}$ is 5-permutable, see Theorem 3.4 in [4]. This immediately implies that for any $\boldsymbol{n} \geq 5$, $\boldsymbol{n}$-permutability is not join-prime in the lattice of interpretability types of varieties.

By following the logic of the proof we sketched here, it is natural to ask if there is a non-3-permutable variety $\mathscr{L}$ such that $\mathscr{L} \vee \mathscr{M}$ is 3-permutable or to raise the analogous question for 4-permutability. In this note, we prove that for any locally finite non-3-permutable variety $\mathscr{L}$, $\mathscr{L} \vee \mathscr{M}$ is not 3-permutable. This points towards the primeness of 3-permutability. However, we expect our 3-permutability result not to hold for non-locally finite varieties. This would imply that there is a non-locally finite non-3-permutable variety whose join with $\mathscr{M}$ is 3-permutable, hence 3-permutability would not be join-prime in the lattice of interpretability types of varieties.

Interestingly, for any non-2-permutable variety $\mathscr{L}$, $\mathscr{L} \vee \mathscr{M}$ is not 2-permutable. We give a proof of this result, independent from Tschantz's unpublished work.

2. PRELIMINARIES

Similarly as in [4], one of our main tools in the proofs is a classical result of Hagemann and Mitschke. In [6], they gave the following characterization of $\boldsymbol{n}$-permutability of a variety.

Theorem 2.1 (Hagemann, Mitschke (1973)). *Let $\mathscr{K}$ be a variety and $\boldsymbol{n} \geq 2$ an integer. Let $\mathbf{F}_2$ be the algebra freely generated by x and y in $\mathscr{K}$. Then the following are equivalent.*

(1) *$\mathscr{K}$ is $\boldsymbol{n}$-permutable.*
(2) *In the digraph whose vertex set is F_2 and whose edge set is the subalgebra generated by $\{(x,x),(x,y),(y,y)\}$ in $\mathbf{F}_2^2$, there is a directed path of length $\boldsymbol{n}-1$ from y to x.*
(3) *Any edge of a reflexive compatible binary relation ρ of any algebra $\mathbf{A} \in \mathscr{K}$ is in a directed $\boldsymbol{n}$-cycle of the digraph (A,ρ).*

A k-ary *compatible relation* of an algebra $\mathbf{A}$ is a subuniverse of $\mathbf{A}^k$. A *compatible relation of a variety* is just a compatible relation of one of the members of the variety.

Let $\mathbb{G}$ denote a reflexive digraph (we allow here $\mathbb{G}$ to be infinite). *The clone of* $\mathbb{G}$ is the set of edge-preserving finitary operations on $\mathbb{G}$. We call an algebra $\mathbb{G}$*-primal* if its underlying set equals the vertex set of $\mathbb{G}$ and its basic operations form a generating set of the clone of $\mathbb{G}$. We denote such an algebra by $\mathbf{G}$ and its base set by G.

Let D be a k-ary relation on G. A *representation of D* is a pair $(\mathbb{R},s)$ where $\mathbb{R}$ is a digraph, $s=(s_0,\dots,s_{k-1})$ is a k-tuple of $\mathbb{R}^k$ such that

$$D=\{\,(f(s_0),\dots,f(s_{k-1}))\mid f\colon \mathbb{R}\to\mathbb{G} \text{ is an edge-preserving map}\,\}.$$

If $(\mathbb{R},s)$ is a representation of D, then we say that D *is defined from* $\mathbb{G}$ by the representation $(\mathbb{R},s)$. It should be clear that D is a compatible relation of $\mathbf{G}$ if D is defined from $\mathbb{G}$ by a representation. We note that every finitary compatible relation of a finite $\mathbf{G}$ has a representation obtained in an obvious way from a primitive positive formula defining the relation in the language of the relation of $\mathbb{G}$ supplemented by the equality, see [3].

Let $D \subseteq G^k$ and ρ a reflexive binary relation on D. Then we conceive ρ as a $2k$-ary relation on G. A representation of ρ is a triple $(\mathbb{R},s,s')$, where $\mathbb{R}$ is a finite digraph,

$$s=(s_0,\dots,s_{k-1}) \text{ and } s'=(s'_0,\dots,s'_{k-1})$$

are two k-tuples of R, and

$$\rho=\{\,((f(s_0),\dots,f(s_{k-1})),(f(s'_0),\dots,f(s'_{k-1})))\mid f\colon \mathbb{R}\to\mathbb{G} \text{ is edge-preserving}\,\}.$$

We say that the relation ρ or the digraph (D,ρ) *is defined from the digraph* $\mathbb{G}$ by the representation $(\mathbb{R},s,s')$, provided that $(\mathbb{R},s,s')$ is representation of ρ. If ρ is defined from $\mathbb{G}$ by the representation $(\mathbb{R},s,s')$, then by the reflexivity of ρ, (R,s) and (R,s') are two representations of D. Thus D is a compatible relation of $\mathbf{G}$. It is also easy to check that ρ is a compatible binary relation of the subalgebra $\mathbf{D}$ determined by D in $\mathbf{G}^k$. As in the remark at the end of the preceding paragraph, if $\mathbb{G}$ is finite, then every reflexive compatible binary relation of a subalgebra of a finite power of $\mathbf{G}$ is defined from $\mathbb{G}$ by some representation.

3. Main results

For a ternary operation symbol m, let $\mathscr{M}$ be the variety defined by the set of majority identities $m(y,x,x) = m(x,y,x) = m(x,x,y) = x$ for m. In this section, we prove the main result of the paper: for any locally finite non-3-permutable variety $\mathscr{L}$, $\mathscr{L} \vee \mathscr{M}$ is not 3-permutable. We also prove that for any non-2-permutable variety $\mathscr{L}$, $\mathscr{L} \vee \mathscr{M}$ is not 2-permutable.

We require some knowledge on the obstructions of a finite digraph $\mathbb{G}$. First we define the notion of a $\mathbb{G}$-obstruction for a digraph $\mathbb{G}$. A $\mathbb{G}$-*colored digraph* is a pair $(\mathbb{H}, f)$ where $\mathbb{H}$ is a digraph and f is a partial map from H to G. A $\mathbb{G}$-colored digraph is *extendible* if there is an edge-preserving total map from $\mathbb{H}$ to $\mathbb{G}$ that extends f. A $\mathbb{G}$-colored digraph $(\mathbb{H}, f)$ is a $\mathbb{G}$-*obstruction* if $\mathbb{H}$ is finite, $(\mathbb{H}, f)$ is non-extendible but for any proper subdigraph $\mathbb{H}'$ of $\mathbb{H}$, $(\mathbb{H}', f|_{H'})$ is extendible.

Clearly, a $\mathbb{G}$-colored digraph $(\mathbb{H}, f)$ is a $\mathbb{G}$-obstruction if and only if $\mathbb{H}$ is finite and connected, $(\mathbb{H}, f)$ is non-extendible but for any edge e, by removing e from $(\mathbb{H}, f)$, the resulting colored digraph is extendible. We note—it should be clear from the definition—that the non-colored vertices of an obstruction span a connected subdigraph of the base digraph of the obstruction. Moreover, if $\mathbb{G}$ is reflexive, then the base digraph of any $\mathbb{G}$-obstruction is an irreflexive digraph. We make use of the following proposition in the later proofs, see Theorem 3.8 in [8].

Proposition 3.1. *A finite digraph $\mathbb{G}$ admits a k-ary near-unanimity operation if and only if the number of colored vertices in any $\mathbb{G}$-obstruction is at most $k-1$.*

The following lemma describes the shape of the $\mathbb{G}$-obstructions in a recursive manner. Earlier, a similar result was obtained for posets, see Proposition 2.3 of [12].

Lemma 3.2. *Let $\mathbb{G}$ be a finite digraph. Let $(\mathbb{H}, f)$ be a $\mathbb{G}$-obstruction. Then for every non-colored vertex h of $(\mathbb{H}, f)$ there exist $\mathbb{G}$-obstructions $(\mathbb{H}_i, f_i)$, $i \in I$, with the following properties.*

1. *For each $i \in I$, $\mathbb{H}_i \subseteq \mathbb{H}$, $h \in H_i$, $f|_{H_i \setminus \{h\}} = f_i|_{H_i \setminus \{h\}}$ and h is colored in $(\mathbb{H}_i, f_i)$.*
2. *For every $p \in G$, there exists $i \in I$ such that if h is recolored by p in $(\mathbb{H}_i, f_i)$, the resulting $\mathbb{G}$-colored digraph is not extendible.*
3. *If h is recolored by $f_j(h)$ in $(\mathbb{H}_i, f_i)$ where $j \in I \setminus \{i\}$, then the resulting $\mathbb{G}$-colored digraph is extendible.*

Proof. If we color h by a vertex p of G in $(\mathbb{H}, f)$, the resulting colored digraph is still non-extendible, so it contains some $\mathbb{G}$-obstructions that must contain h. Let $(\mathbb{H}_t, g_t)$, $t \in T$, be a complete list of pairwise distinct obstructions which can be obtained in this way when p ranges through the vertices of G. For every $t \in T$, let S_t be the subset of vertices of $\mathbb{G}$ by which recoloring h in $(\mathbb{H}_t, g_t)$, the resulting colored digraph is non-extendible. Clearly, $g_t(h) \in S_t$. We choose a subset I of T as follows.

(i) I is a minimal set with respect to $\cup_{i \in I} S_i = G$.
(ii) I has the largest cardinality with respect to (i).
(iii) For every $i \in I$ and $t \in T$ if S_t is a proper subset of S_i, then $S_t \cup (\cup_{j \in I \setminus \{i\}} S_j) \neq G$.
(iv) For every $i \in I$, $\mathbb{H}_i$ is minimal with respect to containment of digraphs among the $\mathbb{H}_t$, $t \in T$, where $S_t = S_i$.

We want to prove that by recoloring h in $(\mathbb{H}_i, g_i)$ for each $i \in I$ in a suitable manner, the resulting $\mathbb{G}$-colored digraphs $(\mathbb{H}_i, f_i)$, $i \in I$ satisfy the claim.

Let $S_i' = S_i \setminus (\cup_{j \in I \setminus \{i\}} S_j)$ for $i \in I$. These sets are nonempty by (i). We claim that every S_i', $i \in I$, contains a vertex p_i such that if h is recolored by p_i in $(\mathbb{H}_i, g_i)$, then resulting colored digraph is an obstruction.

Let us suppose this is not true. So there exists an i such that for every $p \in S_i'$, if h in $(\mathbb{H}_i, g_i)$ is colored by p the resulting colored digraph is not an obstruction. As $p \in S_i$, this colored digraph is non-extendible. Hence it properly contains some obstructions. Let these obstructions be $(\mathbb{H}_v, g_v)$ where $v \in V \subseteq T$, as p ranges through S_i'. Observe that $S_v \subseteq S_i$ and $\cup_{v \in V} S_v \cup (\cup_{j \in I \setminus \{i\}} S_j) = G$. Let V_0 be a minimal subset of V such that $\cup_{v \in V_0} S_v \cup (\cup_{j \in I \setminus \{i\}} S_j) = G$. Clearly, $V_0 \cup (I \setminus \{i\})$ is minimal in the sense of (i). Since $|V_0| \geq 1$ and I satisfies (ii), $V_0 = \{v_0\}$ for some v_0. Then by (iii), $S_i = S_{v_0}$. However, $\mathbb{H}_{v_0}$ is properly contained in $\mathbb{H}_i$, which contradicts the fact that $\mathbb{H}_i$ is minimal by (iv).

Thus, every S_i' contains a vertex p_i such that if h is recolored by p_i in $(\mathbb{H}_i, g_i)$, then the resulting colored digraph denoted by $(\mathbb{H}_i, f_i)$ is an obstruction. Now, the so obtained obstructions $(\mathbb{H}_i, f_i)$, $i \in I$, clearly satisfy the claim of the lemma. $\square$

We remark here that every vertex and edge of $(\mathbb{H}, f)$ must appear in one of the $(\mathbb{H}_i, f_i)$ in the lemma. If this was not true, there would be a vertex or an edge of $(\mathbb{H}, f)$ that is missing from each of the $(\mathbb{H}_i, f_i)$. By removing this vertex or edge from $(\mathbb{H}, f)$, the resulting colored digraph is extendible, say g is an extension of its coloring. Then $g|_{H_i}$ would be an extension of $f|_{H_i \setminus \{h\}} = f_i|_{H_i \setminus \{h\}}$ to $\mathbb{H}_i$, $i \in I$, such that h would have the same color for all $i \in I$ in these extensions, which contradicts item (2) of the lemma.

We also remark that by the use of the lemma, we can construct all $\mathbb{G}$-obstructions of a finite digraph $\mathbb{G}$. Let us suppose we have determined all $\mathbb{G}$-obstructions with fewer than k non-colored vertices. Then any $\mathbb{G}$-obstruction $(\mathbb{H}, f)$ with k non-colored vertices can be obtained by identifying some vertices and edges of a certain $\mathbb{G}$-obstruction $(\mathbb{H}', f')$ that is constructed from $\mathbb{G}$-obstructions with fewer than k non-colored vertices in the following way.

Let $h \in H$ be a non-colored vertex of $(\mathbb{H}, f)$. Let $(\mathbb{H}_i, f_i)$, $i \in I$, be $\mathbb{G}$-obstructions whose existence is guaranteed in the preceding lemma. We take pairwise disjoint copies $(\mathbb{Q}_i, g_i)$ of the $(\mathbb{H}_i, f_i)$, and we denote the copy of h by q_i in $(\mathbb{Q}_i, g_i)$ for every $i \in I$. Then $(\mathbb{Q}_i, g_i)$ has fewer than k non-colored vertices for every $i \in I$. We put together $(\mathbb{H}', f')$ from the $(\mathbb{Q}_i, g_i)$ by deleting the color of q_i in $(\mathbb{Q}_i, g_i)$ for each $i \in I$, and sticking together the resulting colored digraphs at the vertices q_i, $i \in I$.

We claim that the colored digraph $(\mathbb{H}', f')$ constructed from the $(\mathbb{Q}_i, g_i)$ in this way is an obstruction. First, by item (2) of the lemma, $(\mathbb{H}', f')$ is non-extendible. Let e be any edge in $(\mathbb{H}', f')$. We prove that by deleting e in $(\mathbb{H}', f')$, the remaining colored digraph $(\mathbb{H}'', f'')$ is extendible. By the construction, there is a unique i such that e is in $\mathbb{Q}_i$. Since $(\mathbb{Q}_i, g_i)$ is an obstruction that equals a copy of $(\mathbb{H}_i, f_i)$, the $\mathbb{Q}_i$ part in $(\mathbb{H}'', f'')$ is extendible such that the color of q_i is $f_i(h)$. By item (3) of the lemma, the so obtained coloring of the $\mathbb{Q}_i$ part in $(\mathbb{H}'', f'')$ extends to the other copies of $(\mathbb{Q}_j, g_j)$, $j \neq i$, in $(\mathbb{H}'', f'')$. Thus, by deleting any edge in $(\mathbb{H}', f')$, the resulting colored digraph is extendible, which gives that $(\mathbb{H}', f')$ is an obstruction.

Since the $(\mathbb{H}_i, f_i|_{H_i \setminus \{h\}})$ are colored subdigraphs of $(\mathbb{H}, f)$ and $(\mathbb{H}', f')$ is constructed from copies of those, the natural map $\alpha : H' \to H$ that, for each $t \in H$, sends all copies of t from H' to t is an edge-preserving map from $\mathbb{H}'$ to $\mathbb{H}$. Moreover, α maps the colored vertices of $(\mathbb{H}', f')$ to colored vertices of the same color in $(\mathbb{H}, f)$, and α maps the non-colored vertices of $(\mathbb{H}', f')$ to non-colored vertices in $(\mathbb{H}, f)$. By the remark following the lemma, every vertex and edge of $(\mathbb{H}, f)$ has a copy in $(\mathbb{H}', f')$. So the identification of the vertices and edges of $(\mathbb{H}', f')$ by α yields $(\mathbb{H}, f)$.

Next we prove a consequence of Lemma 3.2. This seemingly simple statement will be essential in the proof of the main result of the section. In a digraph, a vertex v is called a *universal out-vertex* if there is an edge from v to each of the vertices of the digraph. A vertex u is called a *universal in-vertex* if there is an edge from each of the vertices of the digraph to u. A vertex of a digraph is called a *sink* if it has out-degree zero. A vertex of a digraph is called a *source* if it has in-degree zero.

Corollary 3.3. *Let $\mathbb{G}$ be a finite reflexive digraph which has a universal out-vertex 0 and a universal in-vertex 1 such that $1 \to 0$ in $\mathbb{G}$. If $\mathbb{G}$ does not admit a majority operation, then there exists a $\mathbb{G}$-obstruction with a single non-colored vertex and at least three colored vertices.*

Proof. By Proposition 3.1, since $\mathbb{G}$ does not admit a majority operation, it has an obstruction with at least three colored vertices. Let $(\mathbb{H}, f)$ be such an obstruction such that the number of its non-colored vertices is the smallest possible. If this number is one, we are done. Suppose that this number is at least two. Let h be any non-colored vertex of $(\mathbb{H}, f)$. Then there exist obstructions $(\mathbb{H}_i, f_i)$, $i \in I$, as in the claim in Lemma 3.2. Since $\mathbb{G}$ is reflexive and each of the $(\mathbb{H}_i, f_i)$ has fewer non-colored vertices than $(\mathbb{H}, f)$, each of them has exactly two colored vertices where one of the colored vertices equals h.

We prove that the colored vertices in the $(\mathbb{H}_i, f_i)$ have degree one. Suppose that u is a colored vertex with degree at least two in $(\mathbb{H}_i, f_i)$ for some $i \in I$. Let e be an edge with end-vertices u and v in $(\mathbb{H}_i, f_i)$. Then v must be a non-colored vertex of $(\mathbb{H}_i, f_i)$. Let e' be a colored copy of e from $(\mathbb{H}_i, f_i)$ such that e' has disjoint end-vertices from H_i. By deleting e in $(\mathbb{H}_i, f_i)$ and adding e' to the remaining colored digraph such that $v \in H_i$ and the copy of v in e' are identified, the resulting colored digraph would be an obstruction with three colored vertices and fewer non-colored vertices than $(\mathbb{H}, f)$, a contradiction.

Now, we prove that each of the $(\mathbb{H}_i, f_i)$ is a directed path of length 1 or 2, colored at the end-vertices. We just proved that the two colored elements of $(\mathbb{H}_i, f_i)$ have degree one. One of the two colored elements must be a source and the other one must be a sink, for otherwise by coloring the non-colored elements in $(\mathbb{H}_i, f_i)$ with 0 or 1, we would get an edge-preserving extension of f_i to $\mathbb{H}_i$. Let $h_0 \to h_1$ and $h_2 \to h_3$ such that h_0 and h_3 are the colored elements of $(\mathbb{H}_i, f_i)$. Then $h_1 = h_2$, for otherwise coloring h_2 by 0 and the other non-colored elements of $(\mathbb{H}_i, f_i)$ by 1, we would get an extension of f_i. Finally, if $h_1 = h_2$, then the only non-colored element of $(\mathbb{H}_i, f_i)$ must be h_1, for otherwise f_i extends to the set $\{h_0, h_1, h_3\}$ by coloring h_1 with some $c \in G$ such that

$$f_i(h_1) \to c \to f_i(h_3),$$

but then, by the reflexivity of $\mathbb{G}$, f_i would be extendible by coloring all non-colored elements of $(\mathbb{H}_i, f_i)$ with c.

Since $(\mathbb{H}, f)$ has at least three colored vertices, each colored vertex of $(\mathbb{H}, f)$ is in one of the $(\mathbb{H}_i, f_i)$ (see the first remark after the preceding lemma), and each of the $(\mathbb{H}_i, f_i)$ contains exactly one colored vertex of $(\mathbb{H}, f)$, $|I| \geq 3$. Just as explained in the second remark following Lemma 3.2, by deleting the color of h in each of the $(\mathbb{H}_i, f_i)$ and gluing together pairwise disjoint copies of the so obtained colored digraphs at the copies of h, we obtain a $\mathbb{G}$-obstruction. So this is an obstruction obtained by gluing together at least 3 pairwise disjoint directed paths of length 1 or length 2 at one of their end-vertices and coloring the other end-vertices in a suitable manner. Among the obstructions of this form, we take one with the smallest number of non-colored vertices, say $(\mathbb{H}', f')$.

We claim that $(\mathbb{H}', f')$ contains only one non-colored vertex. Suppose not. Let h be the common vertex of the directed paths of length 1 and length 2 in the definition of $(\mathbb{H}', f')$. Without loss of generality, we assume that there is a directed path $h_0 \to h_1 \to h$ in $(\mathbb{H}', f')$ where h_0 is a colored vertex of $(\mathbb{H}', f')$. Observe that $(\mathbb{H}', f')$ has an edge (h_2, h) such that h_2 is colored, for otherwise $(\mathbb{H}', f')$ would be extendible by using the colors 0 and 1. An application of the preceding lemma to $(\mathbb{H}', f')$ and its non-colored vertex h_1 together with the remark following the lemma yields that there is a $p \in G$ such that by coloring h_1 with p in $(\mathbb{H}', f')$, the resulting colored digraph contains an obstruction $(\mathbb{H}'', f'')$ such that both h_1 and h_2 belong to $(\mathbb{H}'', f'')$. The obstruction $(\mathbb{H}'', f'')$ must contain a colored vertex apart from h_1 and h_2. Indeed, for otherwise, by using the reflexivity of $\mathbb{G}$, H'' would be equal to the set $\{h, h_1, h_2\}$ and by coloring h with 1, we would obtain an extension of f'' to $\mathbb{H}''$. We also have that $(\mathbb{H}'', f'')$ is assembled from directed paths of length 1 and length 2 by gluing them together at their end-vertex h and that $(\mathbb{H}'', f'')$ has fewer non-colored vertices than $(\mathbb{H}', f')$. So the existence of $(\mathbb{H}'', f'')$ contradicts the definition of $(\mathbb{H}', f')$. $\square$

Let $\mathscr{V}_{\mathbb{P}}$ be the variety generated by an order primal algebra related to the 6-element bounded poset $\mathbb{P}$ that is not a lattice. In Proposition 3.6 of [4], we proved that there is a finite algebra $\mathbf{A}$ that satisfies all identities of $\mathscr{V}_{\mathbb{P}}$, has a majority term operation and the variety generated by $\mathbf{A}$ is not 4-permutable. This implies that the variety $\mathscr{V}_{\mathbb{P}} \vee \mathscr{M}$ is not 4- and hence not 3-permutable. Our main goal in this section is prove that for any locally finite non-3-permutable variety $\mathscr{L}$, $\mathscr{L} \vee \mathscr{M}$ is not 3-permutable.

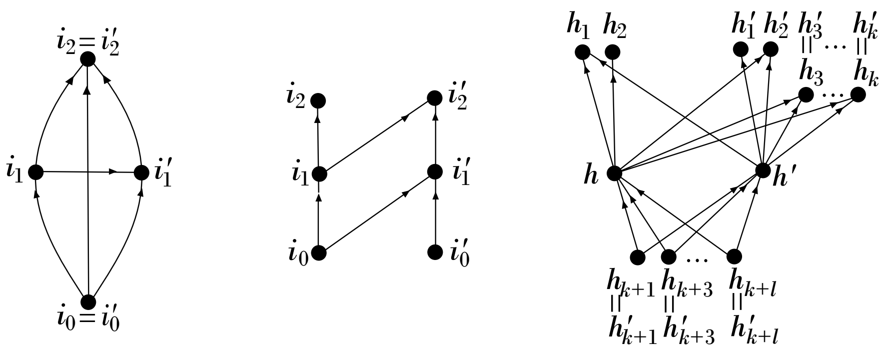

FIGURE 1. Three digraph representations.

For the proof of this result we need some properties of certain special reflexive digraphs defined from an arbitrary reflexive digraph $\mathbb{G}$. The representations of these digraphs are given in Figure 1. In the figure, provided that the relevant representation is $(\mathbb{R}, s, s')$, the tuples s and s' are meant by the naturally ordered lists of the non-primed vertices and primed vertices, respectively. So for the first two items in the figure, $s = (i_0, i_1, i_2)$ and $s' = (i'_0, i'_1, i'_2)$, and for the third item, $s = (h, h_1, h_2, \ldots, h_{k+l})$ and $s' = (h', h'_1, h'_2, \ldots, h'_{k+l})$.

The first item in the figure is called the 01-*representation*, it defines the relation

$$\{((p,q,r),(p,q',r)) :\ p \to q, q' \to r \;\&\; q \to q' \;\&\; p \to r \text{ where } p,q,q',r \in G\}$$

on the set

$$\{(p,q,r) :\ p \to q \to r \;\&\; p \to r \text{ where } p,q,r \in G\}.$$

The second item is called the 010-*representation*, it defines the relation

$$\{((p,q,r),(p',q',r')) :\ p \to q \to r \;\&\; p' \to q' \to r' \;\&\; p \to q' \;\&\; q \to r' \\ \text{where } p,p',q,q',r,r' \in G\}$$

on the set

$$\{(p,q,r) :\ p \to q \to r \text{ where } p,q,r \in G\}.$$

The third item parameterized by k and l where $2 \le k$ and $1 \le l$ is called the *kl-representation*, it defines the relation

$$\{((p,p_1,p_2,p_3,\ldots,p_{k+l}),(p',p'_1,p'_2,p_3,\ldots,p_{k+l})) : \\ p \to p_2 \;\&\; p' \to p'_1 \;\&\; p,p' \to p_1,p'_2,p_i \text{ for } 3 \le i \le k \;\&\; p_i \to p,p' \\ \text{for } k+1 \le i \le k+l \text{ where } p,p',p'_1,p'_2 \in G \text{ and } p_i \in G,\ 1 \le i \le l+k\}$$

on the set

$$\{(p,p_1,p_2,p_3,\ldots,p_{k+l}) :\ p \to p_i \text{ for } 1 \le i \le k \;\&\; p_i \to p \text{ for } k+1 \le i \le k+l \\ \text{where } p \in G \text{ and } p_i \in G,\ 1 \le i \le l+k\}.$$

We also need the notion of $1k$-*representations* when $k \ge 2$. These are the representations obtained from the $k1$-representations by reversing all of the arrows in them.

In a digraph, a vertex v is called a *locally universal out-vertex* if there is an edge from v to each of the vertices of the un-directed component of v. A vertex u is called a *locally universal in-vertex* if there is an edge from each of the vertices of the un-directed component of u to u. We establish some easy properties of the digraphs obtained by the first two representations defined above.

Proposition 3.4. *Let $\mathbb{G}$ be a reflexive digraph. Let $\mathbb{G}_1$ be the digraph defined from $\mathbb{G}$ by the* 01-*representation, and let $\mathbb{G}_2$ be the digraph defined from $\mathbb{G}_1$ by the* 010-*representation. Then the following hold.*

1. *The digraph $\mathbb{G}_1$ is reflexive, and every component of $\mathbb{G}_1$ has a locally universal out-vertex and a locally universal in-vertex.*
2. *The digraph $\mathbb{G}_2$ is reflexive, and every component of $\mathbb{G}_2$ has a locally universal out-vertex and a locally universal in-vertex such that there is an edge from the locally universal in-vertex to the locally universal out-vertex.*

Proof. First we prove item (1). The reflexivity of $\mathbb{G}_1$ is immediately follows from the reflexivity of $\mathbb{G}$. For the other part of item (1), observe that in each component of $\mathbb{G}_1$, (a,a,b) is a locally universal out-vertex and (a,b,b) is a locally universal in-vertex for some edge (a,b) of $\mathbb{G}$. Now we prove item (2). The reflexivity of $\mathbb{G}_2$ is obvious. For the other part of item (2), observe that in each component of

$\mathbb{G}_2$, $(0,0,1)$ is a locally universal out-vertex and $(0,1,1)$ is a locally universal in-vertex for some locally universal out-vertex 0 and in-vertex 1 of a component $\mathbb{G}_1$; moreover, $(0,1,1) \to (0,0,1)$ in $\mathbb{G}_2$. □

Let $\mathbb{G}_0$ be a digraph defined from a digraph $\mathbb{G}$ by the representation $(\mathbb{R}, s, s')$. Let S and S' denote the sets of the entries of the tuples s and s', respectively. Moreover, let $(\mathbb{O}, g)$ be a $\mathbb{G}_0$-colored digraph. We define the *blown-up* $(\hat{\mathbb{O}}, \hat{g})$ *of* $(\mathbb{O}, g)$ *by* $(\mathbb{R}, s, s')$ to be the $\mathbb{G}$-colored digraph as follows.

The digraph $\hat{\mathbb{O}}$ is obtained by taking pairwise disjoint copies $(\mathbb{R}_e, s_e, s'_e)$ of $(\mathbb{R}, s, s)$ for the edges e of $\mathbb{O}$ and gluing these copies together at their S_e and S'_e parts accordingly to the incidence matrix of the digraph $\mathbb{O}$. To make this definition more precise we note that if $e = (u, u')$ and $f = (v, v')$ are different edges in $\mathbb{O}$, and $(\mathbb{R}_e, s_e, s'_e)$ and $(\mathbb{R}_f, s_f, s'_f)$ are the related disjoint copies of $(\mathbb{R}, s, s')$, respectively, then each of the equalities

$$u = v,\ u = v',\ u' = v,\ u' = v'$$

induces the corresponding identification (gluing) from the list

$$s_e \equiv s_f,\ s_e \equiv s'_f,\ s'_e \equiv s_f,\ s'_e \equiv s'_f$$

on the copies $(\mathbb{R}_e, s_e, s'_e)$ and $(\mathbb{R}_f, s_f, s'_f)$ in $\hat{\mathbb{O}}$. Moreover, every loop-edge $e = (u, u)$ induces the identification $s_e \equiv s'_e$.

The coloring $\hat{g}$ of $\hat{\mathbb{O}}$ is given by the natural coloring of the copies of S and S' coming from $(\mathbb{O}, g)$, that is, if $o \in O$ is colored by $a \in G_0$, then the vertices in the copy of S or S' corresponding to o in $\hat{\mathbb{O}}$ are colored by the related entries of the tuple a in the natural way. Thus, if o is the starting vertex of an edge e in $(\mathbb{O}, g)$, $s = (s_0, \dots, s_{k-1})$, and $a = (a_0, \dots, a_{k-1})$, then the copy of s_i in S_e is colored by a_i for each $0 \le i \le k-1$. Similarly, if o is the ending vertex of the edge e in $(\mathbb{O}, g)$ and $s' = (s'_0, \dots, s'_{k-1})$, then the copy of s'_i in S'_e is colored by a_i for each $0 \le i \le k-1$. We remark that $(\mathbb{O}, g)$ is $\mathbb{G}_0$-extendible if and only if $(\hat{\mathbb{O}}, \hat{g})$ is $\mathbb{G}$-extendible.

Now we have all the tools at our disposal to prove our main theorem.

Theorem 3.5. *Let $\mathbb{G}$ be a finite compatible reflexive digraph in a variety $\mathscr{L}$. If $\mathbb{G}$ has an edge (x, y) such that there is no directed path of length 2 from y to x, then there exists a finite compatible reflexive digraph $\mathbb{G}'$ in $\mathscr{L}$ that admits a majority operation and has an edge (x', y') such that there is no directed path of length 2 from y' to x'.*

Proof. We start by defining certain finite reflexive digraphs in $\mathscr{L}$ from $\mathbb{G}$. The first digraph $\mathbb{G}_1$ is defined from $\mathbb{G}$ by the 01-representation. So we know, by Proposition 3.4, that $\mathbb{G}_1$ is a finite reflexive digraph, each of whose components has a locally universal out-vertex and a locally universal in-vertex. Notice that $(x,x,y) \to (x,y,y)$ in $\mathbb{G}_1$, but there is no directed path of length 2 from (x,y,y) to (x,x,y), for otherwise there would be a directed path of length 2 from y to x in $\mathbb{G}$. Moreover, (x,x,y) and (x,y,y) are a locally universal out-vertex and a locally universal in-vertex in the same component of $\mathbb{G}_1$. If $\mathbb{G}_1$ admits a majority operation, we are done. We let $\mathbb{G}' = \mathbb{G}_1$.

If $\mathbb{G}_1$ does not admit a majority operation, then we define $\mathbb{G}_2$ from $\mathbb{G}_1$ by the 010-representation. So, by Proposition 3.4, $\mathbb{G}_2$ is a finite reflexive digraph, each of whose components has a locally universal out-vertex and a locally universal in-vertex such that there is an edge from the locally universal in-vertex to the locally universal out-vertex. Since $\mathbb{G}_1$ has a component with a locally universal out-vertex

u and a locally universal in-vertex v such that there is no directed path of length 2 from v to u, hence $\mathbb{G}_2$ has a component that contains (u,u,u) and (v,v,v) where $(u,u,u) \to (v,v,v)$ and there is no directed path of length 2 from (v,v,v) to (u,u,u). If $\mathbb{G}_2$ admits a majority operation we are done, we let $\mathbb{G}' = \mathbb{G}_2$.

If $\mathbb{G}_2$ does not admit a majority operation, then some component of $\mathbb{G}_2$ does not admit one. Let $\mathbb{C}$ be such a component. We denote by 0 one of the locally universal out-vertices and by 1 one of the locally universal in-vertices in $\mathbb{C}$ such that $1 \to 0$. We invoke Corollary 3.3, so there is a $\mathbb{C}$-obstruction $(\mathbb{H}, f)$ that has a single non-colored vertex h and at least three colored vertices. Let k denote the number of sinks in $\mathbb{H}$, and let l denote the number of sources in $\mathbb{H}$. Since $\mathbb{C}$ has a locally universal out-vertex and a locally universal in-vertex, $1 \leq k, l$, and clearly, $3 \leq k + l$.

Without loss of generality we assume that $2 \leq k$, and we let $\mathbb{G}_3$ be the digraph defined from $\mathbb{G}_2$ by the kl-representation. (If $k = 1$, then, by the use of the $1l$-representation where $l \geq 2$, a dual argument applies.) Clearly, $\mathbb{G}_3$ is a finite reflexive digraph. Suppose that $a_1, \dots, a_k$ are the colors of the sinks and $b_1, \dots, b_l$ are the colors of the sources in $(\mathbb{H}, f)$ such that if we delete the vertex h_i colored by a_i in $(\mathbb{H}, f)$, then h colored by some c_i gives an extension of the remaining colored digraph where $1 \leq i \leq k$. Now observe that

$$(c_1, 1, a_2, \dots a_k, b_1, \dots, b_l) \to (c_2, a_1, 1, \dots a_k, b_1, \dots, b_l)$$

in $\mathbb{G}_3$, as witnessed by the edge-preserving coloring of the first digraph in Figure 2. Moreover, there is no directed path of length 2 from $(c_2, a_1, 1, \dots a_k, b_1, \dots, b_l)$ to $(c_1, 1, a_2, \dots a_k, b_1, \dots, b_l)$ in $\mathbb{G}_3$. This holds since the second colored digraph of Figure 2 contains the obstruction $(\mathbb{H}, f)$, so it is non-extendible.

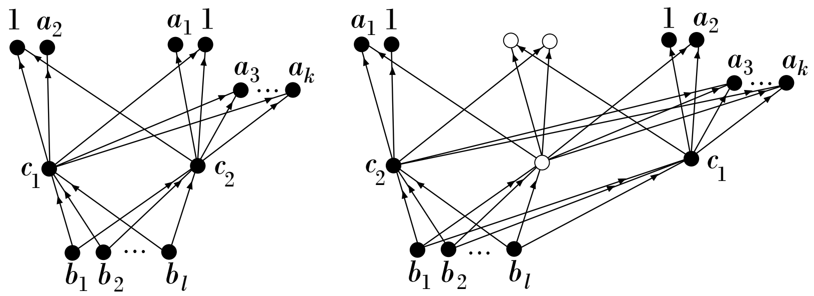

FIGURE 2. An extendible and a non-extendible $\mathbb{G}_3$-colored digraph

Let t be the maximum number of sinks in the $\mathbb{G}_2$-obstructions with a single non-colored vertex. This maximum exists and is finite, since $\mathbb{G}_2$ is finite, and the sinks have no repetition in their colors in any $\mathbb{G}_2$-obstructions with a single non-colored vertex. Since $(\mathbb{H}, f)$ is a $\mathbb{G}_2$-obstruction with a single non-colored vertex and $k \geq 2$, we have that $t \geq 2$. We are going to prove that the number of colored vertices in any $\mathbb{G}_3$-obstruction is at most t, that is, by Proposition 3.1, $\mathbb{G}_3$ admits a $(t+1)$-ary near unanimity operation. As a by-product of the proof, we shall also have that any $\mathbb{G}_3$-obstruction has at most one non-colored vertex, apart from $\mathbb{G}_3$-obstructions with two colored vertices with colors from different components of $\mathbb{G}_3$.

Let $(\mathbb{O}, g)$ be any $\mathbb{G}_3$-obstruction with at least three colored vertices. Then the range of g is contained in one of the components of $\mathbb{G}_3$, for otherwise there would exist a colored subpath of $(\mathbb{O},g)$ whose end-vertices are colored from different components of $\mathbb{G}_3$ and whose other vertices are non-colored, which would contradict to the fact that $(\mathbb{O},g)$ is minimal non-extendible. We want to prove that the number colored vertices in $(\mathbb{O},g)$ is at most t. Let $(\hat{\mathbb{O}},\hat{g})$ be the blown-up of $(\mathbb{O},g)$ by the kl-representation. Clearly, $(\hat{\mathbb{O}},\hat{g})$ is a non-extendible $\mathbb{G}_2$-colored digraph, so it contains a $\mathbb{G}_2$-obstruction $(\mathbb{O}',g')$. The colors that occur in $(\mathbb{O},g)$ are in one component of $\mathbb{G}_3$. Hence, by the definition of $\mathbb{G}_3$, the colors that occur in $(\hat{\mathbb{O}},\hat{g})$ and $(\mathbb{O}',g')$ are in some component $\mathbb{D}$ of $\mathbb{G}_2$.

As $(\mathbb{O},g)$ contains at least one colored vertex and for each $3\le i\le k+l$, all of the copies of h_i from the copies of the kl-representation are identified in $(\hat{\mathbb{O}},\hat{g})$, the only non-colored vertices in $(\hat{\mathbb{O}},\hat{g})$ are some copies of h, h_1 and h_2. The copies of h_1 and h_2 are sinks in $(\hat{\mathbb{O}},\hat{g})$. We prove that neither of the non-colored copies of h_1 and h_2 belong to $(\mathbb{O}',g')$. Let us suppose the contrary. Then by removing any non-colored copy of h_1 or h_2 from $(\mathbb{O}',g')$, we get an extendible $\mathbb{G}_2$-colored digraph. By taking an extension of this colored digraph with coloring the vertex removed with a locally universal in-vertex of $\mathbb{D}$, we would get an edge-preserving $\mathbb{G}_2$-extension of g' to $\mathbb{O}'$, a contradiction. So the non-colored vertices in $(\mathbb{O}',g')$ are copies of h. Since there are no edges between these copies in $(\hat{\mathbb{O}},\hat{g})$, and the digraph of non-colored vertices in $(\mathbb{O}',g')$ is connected, $(\mathbb{O}',g')$ has only one non-colored vertex that is a copy of h. We also remark that $(\mathbb{O}',g')$ must contain colored copies of h_1 or h_2, for otherwise $(\mathbb{O}',g')$ would be extendible by using the entries of the color of any colored element in $(\mathbb{O},g)$. Furthermore, by our assumption, the number of sinks in $(\mathbb{O}',g')$ is at most t.

We define a colored subdigraph $(\mathbb{O}_1,g_1)$ of $(\mathbb{O},g)$. Let z denote the non-colored vertex of $(\mathbb{O}',g')$. For every edge $e'=(z,u)$ of $(\mathbb{O}',g')$ where u is a copy of h_1 or h_2 in $(\mathbb{O}',g')$, there is an edge e in $\mathbb{O}$ such that the copy of the kl-representation related to e (see the definition of the blown-up) contains e'. We pick such an edge e in $(\mathbb{O},g)$ for every edge $e'=(z,u)$ where u is a copy of h_1 or h_2 in $(\mathbb{O}',g')$. One end of the e must be the same non-colored vertex of $(\mathbb{O},g)$, since the blown-ups of the e contain the non-colored element z in $(\mathbb{O}',g')$. The other ends of the e must be colored in $(\mathbb{O},g)$, since the blown-up of an edge e with two non-colored end-vertices does not contain colored copies of h_1 and h_2. Let $(\mathbb{O}_1,g_1)$ be the colored subdigraph spanned by the edges e in $(\mathbb{O},g)$. Clearly, $(\mathbb{O}_1,g_1)$ has a single non-colored vertex. Since exactly one of the vertices of the e are non-colored, the number of colored elements in $(\mathbb{O}_1,g_1)$ is bounded by the number of sinks in $(\mathbb{O}',g')$. So the number of colored vertices in $(\mathbb{O}_1,g_1)$ is at most t.

The blown-up $(\hat{\mathbb{O}}_1,\hat{g}_1)$ of $(\mathbb{O}_1,g_1)$ by the kl-representation obviously contains the edges $e'=(z,u)$ where u is a copy h_1 and h_2 in $(\mathbb{O}',g')$. On the other hand, every edge of the form (v,z) in $(\mathbb{O}',g')$ must be also in $(\hat{\mathbb{O}}_1,\hat{g}_1)$, since the vertex v is a copy of h_i in the kl-representation where $k+1\le i\le k+l$, hence v is in each copy of the kl-representation in $(\hat{\mathbb{O}},\hat{g})$. Moreover, v is colored in $(\hat{\mathbb{O}}_1,\hat{g}_1)$ with the same color as in $(\mathbb{O}',g')$, since $(\mathbb{O}_1,g_1)$ contains colored elements as we noted above. Similarly, (z,w) where w is a copy of h_i in the kl-representation and $3\le i\le k$ must be in $(\hat{\mathbb{O}}_1,\hat{g}_1)$. Then $(\hat{\mathbb{O}}_1,\hat{g}_1)$ is non-extendible, since it contains $(\mathbb{O}',g')$, and hence $(\mathbb{O}_1,g_1)$ is also non-extendible. As $(\mathbb{O},g)$ is minimal non-extendible, $(\mathbb{O},g)=(\mathbb{O}_1,g_1)$. Since the number of colored vertices of $(\mathbb{O}_1,g_1)$ is at most t, the number of colored vertices is also bounded by t in $(\mathbb{O},g)$.

So if $t = 2$, then by Proposition 3.1, $\mathbb{G}_3$ admits a majority operation. Then we are done, we let $\mathbb{G}' = \mathbb{G}_3$. If $t > 2$, we iteratively apply the above three constructions (the 01-representation, the 010-representation, and the kl-representation) to $\mathbb{G}_3$ and the resulting digraphs in an orderly manner. Before finishing the proof, we make two remarks.

Let $\mathbb{P}$ be any finite reflexive digraph such that the number of colored vertices of $\mathbb{P}$-obstructions is at most t'. We first remark that for every finite digraph $\mathbb{Q}$ defined from $\mathbb{P}$ by a representation, the number of colored vertices in any obstruction is at most t'. One way to see this is as follows. By Proposition 3.1, $\mathbb{P}$ admits a near-unanimity operation of arity $t'+1$. Then any finite digraph $\mathbb{Q}$ defined from $\mathbb{P}$ by a representation must also admit a near-unanimity operation of arity $t'+1$. Then again, by Proposition 3.1, the number of the colored vertices in any $\mathbb{Q}$-obstruction is bounded by t'.

We also remark that if $t' > 2$ and $\mathbb{P}$ is a digraph such that every component of $\mathbb{P}$ has a locally universal out-vertex and a locally universal in-vertex, then the maximum number of sinks (and similarly, of sources) in the $\mathbb{P}$-obstructions with a single non-colored vertex is less than t'. This follows from the fact that in such a $\mathbb{P}$-obstruction if the range of its coloring has at least three vertices, then its non-colored vertex is neither a source nor a sink. For otherwise, since there is a component $\mathbb{D}$ of $\mathbb{P}$ that contains the range of the coloring, by coloring the non-colored vertex with a locally universal out-vertex or a locally universal in-vertex from $\mathbb{D}$, we would get an edge-preserving extension of the coloring of the obstruction.

By the first remark, the maximum number of colored vertices of the obstructions of the digraphs obtained in the iterative process is decreasing. If this number, say t', is still greater than 2 after an application the 010-representation when the digraph $\mathbb{P}$ was obtained in the process, then by the second remark, the maximum number t'' of sinks (and similarly, of sources) in the $\mathbb{P}$-obstructions with a single non-colored vertex is less than t'. Let $\mathbb{Q}$ be the digraph obtained from $\mathbb{P}$ by the kl-representation in the process. Similarly, as we proved that for each $\mathbb{G}_3$-obstruction there is a $\mathbb{G}_2$-obstruction with a single non-colored vertex such that the number of colored vertices of the $\mathbb{G}_3$-obstruction is at most the number of sinks of the $\mathbb{G}_2$-obstruction, one obtains that the maximum number of colored vertices of the $\mathbb{Q}$-obstructions is at most t''. So at least in every third step of the process the maximum number of colored vertices of the obstructions is strictly decreasing, and hence eventually, reaches 2. Then the digraph $\mathbb{G}'$ associated with $t' = 2$ in the procedure is a finite compatible reflexive digraph in $\mathscr{L}$ and admits a majority operation. Moreover, $\mathbb{G}'$ also has an edge (x', y') such that there is no directed path of length 2 from y' to x'. □

The following corollary is an immediate consequence of the theorem.

Corollary 3.6. *For every locally finite non-3-permutable variety $\mathscr{L}$, $\mathscr{L} \vee \mathscr{M}$ is non-3-permutable.*

Proof. On the free algebra with the free generators x and y in $\mathscr{L}$, we define a finite reflexive digraph $\mathbb{G}$ whose edge relation ρ is the subalgebra generated by $\{(x,x),(x,y),(y,y)\}$ in the square of the free algebra. By the Hagemann-Mitschke theorem, there is no directed path of length 2 from y to x in $\mathbb{G}$. Then by the preceding theorem, there is a compatible reflexive digraph $\mathbb{G}'$ in $\mathscr{L}$ such that $\mathbb{G}'$ admits a majority operation and has an edge (x', y') such that there is no directed path of length 2 from y' to x'. So $\mathbb{G}'$ is a compatible reflexive digraph in $\mathscr{L} \vee \mathscr{M}$ and, by the Hagemann-Mitschke theorem, $\mathscr{L} \vee \mathscr{M}$ is not 3-permutable. □

We conjecture that there exists a non-locally finite non-3-permutable variety $\mathscr{L}$ such that $\mathscr{L} \vee \mathscr{M}$ is 3-permutable. This conjecture would imply that 3-permutability is not join-prime in the lattice of interpretability types of varieties. Such a conjecture fails for non-2-permutable varieties as shown by the following theorem.

Theorem 3.7. *For every non-2-permutable variety $\mathscr{L}$, $\mathscr{L} \vee \mathscr{M}$ is non-2-permutable.*

Proof. We use the Hagemann-Mitschke theorem for 2-permutability. Let $\mathbb{G}$ be a compatible reflexive digraph in $\mathscr{L}$ such that $\mathbb{G}$ is not symmetric. We shall prove that there exists a compatible reflexive digraph $\mathbb{G}'$ in $\mathscr{L}$ such that $\mathbb{G}'$ is not symmetric and admits a majority operation.

Let $\mathbb{G}_1$ be the digraph defined from $\mathbb{G}$ by the 01-representation. By Proposition 3.4, $\mathbb{G}_1$ is a reflexive digraph, each of whose components has a locally universal out-vertex and a locally universal in-vertex. Since $\mathbb{G}$ is not symmetric, it contains an edge (x,y) such that (y,x) is not an edge of $\mathbb{G}$. Certainly, $((x,x,y),(x,y,y))$ is an edge of $\mathbb{G}_1$ and $((x,y,y),(x,x,y))$ is not an edge of $\mathbb{G}_1$. So $\mathbb{G}_1$ is not symmetric.

Let $\mathbb{H}$ be the 4-element digraph given by the edges

$$i_0 \to i_1,\ i'_0 \to i'_1 \text{ and } i_0 \to i'_1.$$

Let $\mathbb{G}'$ the reflexive digraph defined by the representation $(\mathbb{H},(i_0,i_1),(i'_0,i'_1))$. Notice that for a locally universal out-vertex 0 and a locally universal in-vertex 1 of an arbitrary component of $\mathbb{G}_1$, $(0,1)$ is a locally universal out-vertex and at the same time a locally universal in-vertex in its $\mathbb{G}'$-component. We call such vertices *locally universal vertices*. Clearly, every component of $\mathbb{G}'$ has a locally universal vertex of the above form $(0,1)$.

Let now $\mathbb{C}$ be any component of $\mathbb{G}'$ and u a locally universal vertex in $\mathbb{C}$. Consider the partial majority map defined on the 3-tuples with at most two different components in $\mathbb{C}$. This partial map extends to a fully defined edge-preserving majority operation of $\mathbb{C}$ by giving the value u to every 3-tuple with pairwise different components in $\mathbb{C}$. Thus every component of the digraph $\mathbb{G}'$ admits a majority operation, and hence $\mathbb{G}'$ itself admits one. On the other hand, if (s,t) is an edge in $\mathbb{G}_1$ where (t,s) is not an edge, then $((s,s),(t,t))$ is an edge in $\mathbb{G}'$ and $((t,t),(s,s))$ is not an edge. Hence $\mathbb{G}'$ is not symmetric. So $\mathbb{G}'$ is a compatible reflexive digraph in $\mathscr{L} \vee \mathscr{M}$ and, by the Hagemann-Mitschke theorem, $\mathscr{L} \vee \mathscr{M}$ is not 2-permutable. □

We note that during the editorial process of this paper, by further developing ideas from the proof of the preceding theorem, the three authors gave a short semantic proof of the primeness of 2-permutability in the lattice of interpretability types of varieties. The result has been already published in [5].

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