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A polynomial-time algorithm for near-unanimity graphs

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Abstract

We present a simple polynomial-time algorithm that recognises reflexive, symmetric graphs admitting a near-unanimity operation. Several other characterisations of these graphs are also presented.

Keywords: Near-unanimity operation; Reflexive graph; Polynomial-time algorithm

1. Introduction

The graphs considered here are reflexive and symmetric, i.e., finite relational structures (G, θ) where θ is a binary relation on G satisfying $(x, x) \in \theta$ for all $x \in G$ and $(y, x) \in \theta$ whenever $(x, y) \in \theta$. The elements of G are called *vertices*. We usually write xy to mean $(x, y) \in \theta$ and say that xy is an *edge* of the graph G or that x and y are *adjacent* in G.

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We will also refer to x as the neighbour of y, and vice versa. A vertex with only itself as a neighbour is an *isolated vertex*. The set of all neighbours of x is called the *neighbourhood* of x and is denoted by $N(x)$. A graph H is a *subgraph* of a graph G if $H \subseteq G$ and each edge of H is also an edge of G; H is an *induced subgraph* of G for all vertices $x, y \in H$ such that xy is an edge of G, xy is also an edge of H. A *path of length* k in a graph G is a sequence of vertices $x_0, x_1, \dots, x_k$ such that $x_i x_{i+1}$ is an edge of G for $i = 0, \dots, k-1$; note that we do not require that the vertices be distinct. A graph is *connected* if there exists a path between all pairs of vertices and *disconnected* otherwise. A *component* of a graph G is an induced subgraph H of G that is maximal with respect to being connected; thus for a disconnected graph G, if R is a subgraph of G and H is a proper subgraph of R, then R is disconnected.

The product we consider here is the usual product of structures, namely, if G and H are graphs, then (g_1, h_1) and (g_2, h_2) are adjacent in $G \times H$ if g_1 is adjacent to g_2 in G and h_1 is adjacent to h_2 in H. An *k-ary operation* on a graph G is a graph homomorphism $f : G^k \to G$. If f is an operation on G we say that G *admits* f, or that f is *compatible* with G. An operation is *idempotent* if it satisfies the identity $f(x, x, \dots, x) \approx x$ (i.e., $f(x, x, \dots, x) = x$ for all $x \in G$). An operation f is a *near-unanimity operation* if f is idempotent and $f(y, x, \dots, x) \approx f(x, y, x, \dots, x) \approx f(x, x, \dots, x, y) \approx x$.

Let G be a graph. We call a vertex x of G *dismantlable* if there exists a distinct vertex y of G such that $N(x) \subseteq N(y)$; we say that the vertex y *dominates* x. Note that as G is reflexive, x must be in $N(y)$ and so x and y are adjacent. A graph G is *dismantlable* if we can write G as a sequence of vertices $x_1, \dots, x_n$ such that x_i is dismantlable in the subgraph of G induced by $\{x_i, \dots, x_n\}$, $i = 1, \dots, n-1$. Such a sequence is a *dismantling ordering*. Observe that if a graph G is dismantlable, it must also be connected; either G has only one vertex or G has a dismantling ordering $x_1, \dots, x_n$, where for each $i < n$, x_i has a neighbour x_j, $i < j \leqslant n$. Thus, if x_k is the vertex of highest index in a component of G, either $x_k = x_n$ or x_k must have a neighbour x_p, $k < p \leqslant n$. Therefore G can only have one component. Let H be an induced subgraph of G. If there exists a sequence of vertices $x_1, \dots, x_k$ in G such that $H = G \setminus \{x_1, \dots, x_k\}$ and x_i is dismantlable in the subgraph of G induced by $\{x_i, \dots, x_k\} \cup H$, then we say that G *dismantles* to H and that $x_1, \dots, x_k$ is a *dismantling* to H. We call G *ramified* if G has at least two vertices and has no dismantlable vertices.

Let G be a graph. An induced subgraph X of the graph G^k is an *idempotent k-subalgebra* if it is invariant under all idempotent operations on G. More precisely, let f be an n-ary idempotent operation on G and let X be a subset of G^k. Then X is *invariant under* f if, for any matrix M of size $k \times n$ whose columns are in X, the column obtained by applying f to the k rows of M is also in X.

Given two graphs G and H, $\mathrm{Hom}(H, G)$ is the graph whose vertex set is all homomorphisms from H to G, where homomorphisms f and g are adjacent in $\mathrm{Hom}(H, G)$ if $f(x)g(y)$ is an edge of G whenever xy is an edge of H. Note that $\mathrm{Hom}(H, G)$ is also reflexive, and that composition is edge-preserving. A *constant map* from H to G is map that sends all vertices of H to one particular vertex of G. For a given vertex x of G, we denote by $\bar{x}$ the map that sends all of H to x. If G is connected, then the constant maps from H to G are in the same component of $\mathrm{Hom}(H, G)$. Denote this component by $C_{H,G}$.

Near-unanimity operations have attracted a great deal of attention in recent years, not only in universal algebra [1,3,8,16,25], but also in the context of graph theory [2,4,10, 15,22] and computer science. For instance, a (restricted) constraint satisfaction problem (CSP) is of bounded strict width if and only if its target structure admits a compatible near-unanimity operation, if and only if the target structure satisfies the l-Helly property for some l [12]. In particular, it is well known that a CSP whose target structure admits a compatible near-unanimity operation is solvable in polynomial-time [14] (see also [9]). However, it is still not known if the property of admitting a compatible near-unanimity operation is actually decidable [12]. The closely related problem of determining whether a finite algebra admits a near-unanimity term operation is thought to be undecidable, and although partial results have been obtained the question remains open [23].

In view of the above, it seems natural to seek out classes of structures for which the problem of determining the existence of compatible near-unanimity operations is decidable, or even tractable. Families of finite algebras for which the problem is decidable are presented in [23], and a polynomial-time algorithm to recognise finite posets with a compatible near-unanimity operation can be found in [17], based on a characterisation of these posets by the first and third authors [20]. Reflexive binary structures are a natural generalisation of posets, and much more flexible. For instance, these can be used as a tool for showing some non-trivial **NP**-completeness results (see [18,19]). The present paper describes a characterisation of reflexive, symmetric graphs admitting a compatible near-unanimity operation from which we will easily deduce a polynomial-time algorithm to recognise them:

Theorem 1.1. *A finite, reflexive, symmetric graph admits a near-unanimity operation if and only if each of its connected components does. A connected graph G admits a near-unanimity operation if and only if G^2 dismantles to the diagonal.*

The procedure for the connected case is as follows: look in $G^2 \setminus \Delta$ for a dismantlable vertex, where Δ denotes the diagonal $\Delta = \{(x,x)\colon\ x \in G\}$. If there is none, stop. Otherwise, remove such a vertex u, and repeat the procedure with $G^2 \setminus \{u\}$. When the procedure stops, there are two possibilities: if we have reached Δ, then G admits a near-unanimity operation; otherwise, we obtain an induced subgraph K of G^2 which does not dismantle to the diagonal. By Corollary 2.6, this implies that G^2 does not dismantle to the diagonal so G does not admit a near-unanimity operation.

The characterisation of reflexive, symmetric graphs with a compatible near-unanimity operation (Theorem 3.1) has both an algebraic and a combinatorial flavour, and is similar in spirit to Theorem 4.1 in [5] concerning dismantlable graphs. In fact, our characterisation has some interesting finite model-theoretic consequences: fix a finite structure $\mathcal{T}$, and consider the *retraction problem* for $\mathcal{T}$: given a structure $\mathcal{S}$ similar to $\mathcal{T}$ that contains it as a substructure, determine if there is a retraction of $\mathcal{S}$ onto $\mathcal{T}$ (see for instance [6,7]). V. Dalmau, A. Krokhin and the first author have recently shown, using the characterisation found here, that the reflexive, symmetric graphs whose retraction problem is first-order definable are precisely those that are connected and admit a compatible near-unanimity operation [6]. A similar result holds for posets, but only if the inputs are restricted to posets; in fact, it is also noted in the same paper that the analogous result cannot hold for digraph retraction problems.

Furthermore, the results for posets and symmetric graphs do not seem to generalise easily to general reflexive digraphs. Indeed, one of the keys to the existence of a polynomial-time algorithm to recognise symmetric graphs (or posets) with a near-unanimity operation is found in Lemma 2.5 below. Let G be a symmetric graph or a poset, and let r be a retraction of G onto a subgraph H. If there exists a path in $\mathrm{Hom}(G, G)$ consisting of retractions onto subgraphs containing H from the identity map to r, then G can be dismantled to H. Since the restriction of a retraction to its image is the identity, and composition of maps preserves adjacency, one may recursively reduce to the following situation: if G retracts to H via a retraction r such that (id, r) is an edge in $\mathrm{Hom}(G, G)$ then there exists a path of retractions from the identity to r, in such a way that the image of each is obtained by removal of one element of the preceding one. This provides a straightforward procedure to find such retracts. However it is easy to come up with counterexamples in the case of general digraphs: let $f : A \to B$ be a homomorphism from the (reflexive) digraph A to the digraph B. Define a new digraph C as follows: it is the disjoint union of A and B, with the extra edges $(a, f(a'))$ for every edge (a, a') in A. It is easy to verify that the map r defined by $r(t) = t$ if $t \in B$ and $r(t) = f(t)$ if $t \in A$ is an edge-preserving retraction of C onto B such that (id, r) is an edge in $\mathrm{Hom}(C, C)$. By choosing for instance A and B to be the same oriented cycle of length at least 3, and f the identity, we obtain a digraph which is domination-free.

Problem 1.2. Is there a polynomial-time algorithm that decides, given a digraph G and an induced subdigraph H of G, whether there exists a retraction r of G onto H such that either (id, r) or (r, id) is an edge of $\mathrm{Hom}(G, G)$?

It should also be noted that in fact even some of the results for posets do not carry over to the symmetric graph case: in [17], it is shown that if in a poset P every idempotent 1-subalgebra is connected (dismantlable) then P admits a near-unanimity operation. This does not hold in the case of graphs:

Proposition 1.3. *The graph G pictured in Fig.* 1 *admits no near-unanimity operation but all its idempotent* 1*-subalgebras are dismantlable.*

Proof. The fact that this graph admits no near-unanimity operation follows from our algorithm: the diagonal of G^2 does not dismantle to the diagonal. An alternative proof goes as follows: by a result of Feder and Hell [11], if a graph has a compatible near-unanimity operation then it must be an absolute retract for arc-consistency and it is known that the graph G does not have this property [22].

We now show that every idempotent 1-subalgebra of G is dismantlable. Let $0'$, 0, $1'$, 1, $2'$, 2, $3'$, 3 denote the vertices of G on the outside cycle, starting from the top left corner; let u denote the middle vertex.

Let X be an idempotent 1-subalgebra of G with at least 2 elements. We use the following equivalent description of idempotent 1-subalgebras (see, for example, [27]): there exists a triple (H, f, h_0) where H is a reflexive, symmetric graph, f is a partial map from H to G

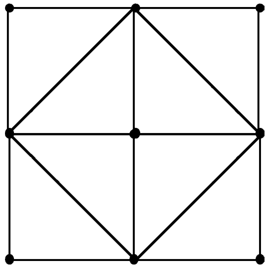

Fig. 1. A graph with no nuf whose subalgebras are all dismantlable.

and $h_0 \in H$, such that X consists of all $g(h_0)$ where g is a homomorphism from H to G extending f.

Claim 1. *If one of $x \in \{0', 1', 2', 3'\}$ is in X then so are its neighbours.*

Indeed, if $g(h_0) = x$ where g extends the map f, and y dominates x in G, then the map g' defined by $g'(t) = y$ if $t = h_0$ and $g'(t) = g(t)$ otherwise clearly extends f and is a homomorphism; since any $x \in \{0', 1', 2', 3'\}$ is dominated by its neighbours, this proves our claim.

Claim 2. *If X contains $\{0, 1, 2, 3\}$ then it contains u.*

Consider the self-map α of G defined by $\alpha(i') = i$ for all $i = 0, 1, 2, 3$ and $\alpha(x) = x$ otherwise. It is clear that α is a homomorphism, and one verifies easily that α is adjacent to id in $\mathrm{Hom}(G, G)$. Now define a 4-ary idempotent operation ϕ on G as follows:

$$\phi(x, y, z, w) = \begin{cases} u, & \text{if } (x, y, z, w) = (0, 1, 2, 3), \\ x, & \text{if } x = y = z = w, \\ \alpha(x), & \text{otherwise.} \end{cases}$$

A straightforward verification shows ϕ is a homomorphism. It follows that X is closed under ϕ. Thus, if ϕ contains $\{0, 1, 2, 3\}$, it also contains u.

So suppose that X contains a member of $\{0', 1', 2', 3'\}$; by Claim 1 we see that X must be connected; and that it dismantles to its intersection with $\{0, 1, 2, 3, u\}$. By direct inspection and Claim 2 we conclude that $X \cap \{0, 1, 2, 3, u\}$ is itself dismantlable and hence so is X.

We may now assume that $X \subseteq \{0, 1, 2, 3, u\}$. Inspection shows that either X is dismantlable, or without loss of generality (by symmetry of G) we have $X = \{0, 2\}$. We prove that this last case is not possible. Indeed, $0 \in X$ means that there is a homomorphism g from H to G extending f such that $g(h_0) = 0$. Define a map $g' : H \to H$ as follows:

$$g'(t) = \begin{cases} u, & \text{if } t = h_0, \\ f(t), & \text{if } t \text{ is in the domain of } f, \\ \alpha(g(t)), & \text{otherwise.} \end{cases}$$

We claim that g' is a homomorphism. Indeed, because α and id are adjacent, the only possible problem that might arise is if some neighbour x of h_0 is such that $f(x) = 0'$ or

$f(x) = 1'$. But since $2 \in X$ there exists a homomorphism from H to G mapping h_0 to 2 that extends f, and so this is not possible. Hence we conclude that $u \in X$, a contradiction. □

A few words of explanation are in order concerning the universal algebraic aspects of our characterisation. To each graph (or more generally, to each relational structure) one may associate naturally a (universal) algebra $\mathbb{A}$ with universe the set of vertices of the graph, and whose n-ary basic operations are all the homomorphisms from G^n to G. The nature of the identities satisfied by the operations of this algebra has important repercussions on the shape of the graph (see, for example, [21].) The interest in so-called *Gumm terms* and *Jónsson terms* (see Theorem 3.1) stems from the fact that they characterise those equational classes that are respectively congruence-modular and congruence-distributive (see [13] or [24].)

For the sake of completeness, we have included a full proof of Theorem 3.1 even though some of the steps may be obtained with few changes from the poset case, see [20]. The next section contains the basic results about graph obstructions, graph homomorphisms and identities on graphs we shall require in the proof of Theorem 3.1, and Section 3 contains the proof of the two main results, Theorems 3.1 and 1.1.

2. Preliminaries

2.1. Obstructions

To prove the existence of our intended algorithm, we will be exploiting the very strong relationship between near-unanimity operations and partial mappings that do not extend. An *obstruction* for the graph G is a pair (H, f) where H is a graph and f is a partial map from H to G such that

(1) the map f does not extend to a full homomorphism from H to G, and
(2) (H, f) is minimal with respect to property (1).

Minimality here is in the following sense: we say that the pair $(H', f') \subseteq (H, f)$ if H' is a subgraph of H and f' is the restriction of f to H'. It is easy to see that H must be connected. We call the vertices of H for which f is defined *coloured* and we call f a *homomorphism* if it is a homomorphism on the coloured vertices of H. Obstructions were first defined by Zádori for general relational structures [29], generalising his *zig-zags* on posets [30].

The following lemma was first pointed out in the poset form in [28] without proof. The proof can be found in Lemma 1.17 [29], the slight change in the definition of obstruction we use does not affect it.

Lemma 2.1. *Let $k \geqslant 3$. A graph G admits a k-ary near-unanimity operation if and only if in every obstruction (H, f) on G the number of coloured vertices is at most $k - 1$.*

Lemma 2.2. *Let (H, f) be an obstruction on a graph G where f is not a homomorphism. Then $H = \{x, y\}$, where xy is an edge of H and $f(x)f(y)$ is not an edge of G.*

Proof. As f is not a homomorphism, there exists $x, y \in H$ such that xy is an edge of H, and $f(x)f(y)$ is not an edge of G. Suppose that there exits a vertex z in H distinct from x and y. Then $(H \setminus z, f|_{H\setminus z})$ is still not extendible, contradicting the minimality of (H, f). Hence $H = \{x, y\}$ and the lemma is true. □

Lemma 2.3. *Let (H, f) be an obstruction on a graph G where f is a partial homomorphism. Let xy be an edge of H, $x \neq y$. Then if x is coloured, y is not.*

Proof. Suppose that both x and y are coloured. Let H' be the graph we obtain by removing the edge xy from H. By the definition of an obstruction, there exists an extension f' of (H', f). But, since f was a homomorphism, $f'(x)f'(y)$ must be an edge of G, and so f' is an extension of (H, f), contradiction. □

Lemma 2.4. *Let (H, f) be an obstruction on G. If a vertex x of H has distinct coloured neighbours z and y, then $f(y) \neq f(z)$.*

Proof. If f is not a homomorphism then the claim is clear by Lemma 2.2. Otherwise, by Lemma 2.3, x can not be coloured. Let H' be the graph we obtain by removing the edge xy from H. Consider (H', f). By the definition of obstructions, there exists an extension f' of (H', f). If $f(y) = f(z)$, then we must also have $f'(y) = f'(z)$. Since $f'(x)f'(z)$ is an edge of G, f' must also be an extension of (H, f), contradiction. □

2.2. Dismantlable and ramified graphs

Lemma 2.5. *Let G be a graph and H an induced subgraph of G. Then the following statements are equivalent*:

(1) *G dismantles to H.*
(2) *There exists a path $\{f_i\}$ $(i = 0, \ldots, m)$ in $\mathrm{Hom}(G, G)$ such that*
  (i) $f_0 = \mathrm{id}$,
  (ii) *f_m is a retraction onto H, and*
  (iii) *$H \subseteq f_i(G)$ for all $i = 0, \ldots, m$.*

Proof. (1) ⇒ (2). Simply notice that if y dominates x in G then the retraction that sends x to y and fixes all other vertices is adjacent to the identity in $\mathrm{Hom}(G, G)$, so we may use induction on $|G| - |H|$: if $G = H$ the result is obvious. Now let r be the retraction described above of G onto $G' = G \setminus \{x\}$. By induction, there exists a path $\{f_i'\}$ $(i = 0, \ldots, m)$ in $\mathrm{Hom}(G', G')$ satisfying (ii). It is clear that $\{\mathrm{id}\} \cup \{f_i' \circ r\}$ is a path in $\mathrm{Hom}(G, G)$ that satisfies the desired conditions.

(2) ⇒ (1). Let $\{f_i\}$ $(i = 0, \ldots, m)$ in $\mathrm{Hom}(G, G)$ be a path that satisfies the desired conditions. We may in fact assume that each f_i is a retraction: indeed we may replace f_i by f_i^s where s is large enough so that f_i^s is a retraction for all $i = 0, \ldots, m$; since composition

is edge-preserving the resulting sequence has all the desired properties. We may also suppose that $f_i(G) \supseteq f_j(G)$ for all $i \leqslant j$. To see this, compose f_1 on the left with all the retractions following f_1 in the path: one obtains the new path id, $f_1, f_1 \circ f_2, \ldots, f_1 \circ f_m$ which still has the desired properties and is such that the image of every map that follows f_1 in the sequence is contained in $f_1(G)$. Again we may iterate each map to obtain retractions, say a path $\{g_i\}$. Repeat the procedure, this time starting at g_2: one obtains the path id, $g_1, g_2, g_2 \circ g_3, \ldots, g_2 \circ g_m$. Repeating this way we obtain the desired path.

To finish the proof, it is now sufficient to consider the following situation: let r be a retraction of G onto R which is adjacent to the identity in $\mathrm{Hom}(G, G)$; we claim that G dismantles to R. Indeed, if $G = R$ we are done. Otherwise, let $x \notin R$. If xy is an edge of G then $r(x)y$ is an edge also since (id, r) is an edge of $\mathrm{Hom}(G, G)$. Hence $N(x) \subseteq N(r(x))$ and so G dismantles to $G \setminus \{x\}$. Obviously the restriction of r to $G \setminus \{x\}$ is adjacent to the identity so by induction, $G \setminus \{x\}$ dismantles to R hence we are done. Combining this with the path above yields the result. □

Corollary 2.6. *Let $H \subseteq K$ be induced subgraphs of the graph G. If G dismantles to H then K dismantles to H.*

Proof. Suppose G dismantles to H. Restrict to K the path whose existence is guaranteed by the last lemma; it is easy to see that this is a path in $\mathrm{Hom}(K, K)$ with similar properties, and so by Lemma 2.5, K dismantles to H. □

Lemma 2.7. *Let G be a graph. Then the following statements are equivalent*:

(1) *G is ramified.*
(2) *The identity function is an isolated vertex in* $\mathrm{Hom}(G, G)$.
(3) *Each automorphism of G is an isolated vertex in* $\mathrm{Hom}(G, G)$.

Proof. (1) ⇒ (2). Let G be a ramified graph. Let f be a vertex of $\mathrm{Hom}(G, G)$ that is adjacent to id_G. Let x be a vertex of G. Then $\mathrm{id}_G(z)f(x)$ is an edge of G for all neighbours z of x in G. As G is ramified, $f(x) = x$. Therefore $f = \mathrm{id}_G$.

(2) ⇒ (3). Let G be a graph such that id_G is an isolated vertex in $\mathrm{Hom}(G, G)$. Let f be an automorphism on G. Suppose that f is adjacent to g in $\mathrm{Hom}(G, G)$. Thus $f(x)g(y)$ is an edge of G whenever xy is an edge of G. This implies that $\mathrm{id}_G = f \circ f^{-1}$ and $g \circ f^{-1}$ are adjacent in $\mathrm{Hom}(G, G)$. Therefore $g \circ f^{-1} = \mathrm{id}_G$, and so $f = g$. Therefore each automorphism on G is an isolated vertex in $\mathrm{Hom}(G, G)$.

(3) ⇒ (1). Immediate by Lemma 2.5. □

Lemma 2.8. *Let G be a connected graph. Then the following statements are equivalent*:

(1) *G is dismantlable.*
(2) *No retract of G is ramified.*
(3) $\mathrm{Hom}(H, G)$ *is connected for every graph H.*

Proof. (1) $\Rightarrow$ (2). Let r be a retraction of G onto H, and let $\{f_i\}$ be a path in $\mathrm{Hom}(G, G)$ from the identity to a constant map. Then $\{r \circ f_i|_H\}$ is a path in $\mathrm{Hom}(H, H)$ from the identity to a constant function; hence by Lemma 2.7, H cannot be ramified.

(2) $\Rightarrow$ (1). Trivial.

(1) $\Rightarrow$ (3). Let $f \in \mathrm{Hom}(H, G)$ and let $\{f_i\}$ be a path in $\mathrm{Hom}(G, G)$ from the identity to a constant map. Then the compositions $f_i \circ f$ form a path from f to a constant map in $\mathrm{Hom}(H, G)$.

(3) $\Rightarrow$ (1). By hypothesis there exists a path $\{f_i\}$ in $\mathrm{Hom}(G, G)$ with $f_0 = \mathrm{id}$ and f_m a constant map. As we did in the proof of Lemma 2.5, it is easy to compose these maps in the right way to ensure that the value of the constant map f_m will lie in $f_i(G)$ for all i; by Lemma 2.5 this shows that G is dismantlable. □

Lemma 2.9. *Let G and H be graphs, where G is connected. Let $f \in \mathrm{Hom}(H, G)$. If there exists a dismantlable induced subgraph X of G such that $f(H) \subseteq X$, then f is in $C_{H,G}$.*

Proof. The graph $\mathrm{Hom}(H, X)$ is connected by Lemma 2.8 and so there exists a path from f to some constant map in $\mathrm{Hom}(H, X)$. As $\mathrm{Hom}(H, X)$ is an induced subgraph of $\mathrm{Hom}(H, G)$, there is a path from f to a constant map in $\mathrm{Hom}(H, G)$. □

2.3. Identities

In this section we state a result we shall require in the proof of the main result. Since the proof is a straightforward variant of the poset case (Lemma 4.1 and Theorem 4.2 of [20]), we omit it.

A graph G admits *Gumm operations* (see [24]) if there exist 3-ary operations $d_0, \dots, d_n, p$ on G that satisfy:

$$d_0(x, y, z) \approx x, \tag{1}$$

$$d_i(x, y, x) \approx x \quad \text{for all } i, \tag{2}$$

$$d_i(x, x, y) \approx d_{i+1}(x, x, y) \quad \text{for } i \text{ even}, \tag{3}$$

$$d_i(x, y, y) \approx d_{i+1}(x, y, y) \quad \text{for } i \text{ odd}, \tag{4}$$

$$d_n(x, y, y) \approx p(x, y, y), \tag{5}$$

$$p(x, x, y) \approx y. \tag{6}$$

Theorem 2.10. *Let G be a connected graph that admits Gumm operations. Then G is dismantlable.*

3. Main results

The next theorem is the analogue of Theorem 4.3 in [20]. The proof closely follows that of Theorem 4.3, with the exception of the implication that bounded diameter of obstructions implies a finite number of obstructions; this is a modification of a construction used in [30] for posets.

Theorem 3.1. *Let G be a finite, connected graph. Then the following conditions are equivalent*:

(1) *G admits a near-unanimity operation.*
(2) *G admits Jónsson operations.*
(3) *G admits Gumm operations.*
(4) *For every $k \geqslant 1$, every idempotent k-subalgebra of G is dismantlable.*
(5) *For every $k \geqslant 1$, every idempotent k-subalgebra of G is connected.*
(6) *There is a path of idempotents between the 2 projections in* $\mathrm{Hom}(G^2, G)$.
(7) *The diameter of obstructions for G is bounded.*
(8) *The number of obstructions for G is finite.*

Proof. The proof that (1) implies (2) implies (3) is immediate by standard results in universal algebra (see [24]).

(3) $\Rightarrow$ (4). Let H be an idempotent k-subalgebra. Since G admits Gumm operations $d_0, \ldots, d_n, p$, which are all idempotent, H also admits Gumm operations. By Theorem 2.10, if H is connected, then H is dismantlable. Therefore all we need to do is prove that H is connected.

Let x and y be two vertices of H; we will show that there is a path between x and y in H. By Theorem 2.10, G is dismantlable and so $\mathrm{Hom}(G^2, G)$ is connected by Lemma 2.8. For $i = 0, \ldots, n$, define maps

$$\Phi_i : \mathrm{Hom}\big(G^2, G\big) \to H$$

by

$$\Phi_i(f) = d_i\big(x, f(x, y), y\big)$$

for all $f \in \mathrm{Hom}(G^2, G)$. We claim that Φ_i is a well defined homomorphism, $i = 0, \ldots, n$. Clearly, Φ_i is a homomorphism as d_i and f are both homomorphisms, $i = 0, \ldots, n$. Now all we need to do is prove that the image of $\mathrm{Hom}(G^2, G)$ under Φ_i is contained in H. Note that the function $\Phi_i(f)$ is equal to the function $d_i(\pi_1, f, \pi_2)$ evaluated at (x, y), where π_1 is the projection on the first variable and π_2 the projection on the second. By identity (2), $d_i(\pi_1, f, \pi_2)$ is idempotent. Hence as $x, y \in H$, $\Phi_i(f)$ is also in H.

Since Φ_i is a homomorphism and since $\mathrm{Hom}(G^2, G)$ is connected, $\Phi_i(\mathrm{Hom}(G^2, G))$ is a connected subgraph of H for $i = 0, \ldots, n$. For i even, identity (3) implies that

$$\Phi_i(\pi_1) = d_i(x, x, y) = d_{i+1}(x, x, y) = \Phi_{i+1}(\pi_1)$$

and for i odd, identity (4) implies that

$$\Phi_i(\pi_2) = d_i(x, y, y) = d_{i+1}(x, y, y) = \Phi_{i+1}(\pi_2).$$

Therefore the images of Φ_i and Φ_{i+1} intersect for all $i = 0, \ldots, n$. Let K be the component of H that contains these images. By identity (1), $\Phi_0(f) = d_0(x, f(x, y), y) = x$ for all $f \in \mathrm{Hom}(G^2, G)$, and so $x \in K$. By identity (5), $\Phi_n(\pi_2) = d_n(x, y, y) = p(x, y, y)$, and so $p(x, y, y)$ is in K. All we need to do now is prove that y is also in K.

Since G^k is connected, there exists a shortest path from y to some vertex z of K, say of length m. As G^k is reflexive, there exists a path of length $2l \leqslant m + 1$ from y to z; if m is odd, create a path of length $m+1$ by repeating the vertex z. Let $y = w_0, w_1, \dots, w_{2l} = z$ be such a path. By applying the above argument, $p(z, y, y) \in K$. Now consider the path $p(w_{2l}, w_0, y), p(w_{2l-1}, w_1, y), \dots, p(w_l, w_l, y)$ in G^k. As $p(w_l, w_l, y) = y$ by identity (6) and as $p(w_{2l}, w_0, y) = p(z, y, y)$, this is a path of length l from y to a vertex of K and so $l \leqslant m$. Therefore $m \leqslant 1$, and so yz is an edge of G^k. Since $z, y \in H$ and H is an induced subgraph of G^k, zy is also an edge of H. Therefore $y \in K$.

(4) $\Rightarrow$ (5). This is obvious by the definition of dismantlability.

(5) $\Rightarrow$ (6). Is trivial, since $\mathrm{Hom}(G^2, G)$ is a subset of the graph $G^{|G|^2}$, and the set of idempotent binary operations constitutes an idempotent $|G|^2$-subalgebra.

(6) $\Rightarrow$ (7). Let P denote a path of length m between the two projections consisting of idempotent operations in $\mathrm{Hom}(G^2, G)$. Suppose for a contradiction that there exists an obstruction (H, f) for G of diameter $n \geqslant m+1$: let h_1, h_2 be vertices of H that are at distance n from one another. Since paths are absolute retracts for reflexive graphs [26], it is easy to see that there exists a homomorphism $\phi: H \to P$ such that $\phi(h_i) = \pi_i$ for $i = 1, 2$. Let f_i be an extension of f when h_i is removed from H. Then we define a map F from H to G as follows:

$$F(h) = \begin{cases} f_2(h_1) & \text{if } h = h_1, \\ f_1(h_2) & \text{if } h = h_2, \\ \phi(h)(f_2(h), f_1(h)) & \text{otherwise.} \end{cases}$$

It is clear that F extends f since each f_i extends f and $\phi(h)$ is idempotent for all h. We claim that F is a homomorphism. Indeed, suppose that u and v are adjacent in H. Without loss of generality the only case we need to consider is when $u = h_1$ and $v \neq h_1$: then

$$F(h_1) = f_2(h_1) = \pi_1\big(f_2(h_1), f_1(v)\big) = \phi(h_1)\big(f_2(h_1), f_1(v)\big)$$

is adjacent to

$$\phi(v)\big(f_2(v), f_1(v)\big) = F(v)$$

and we are done.

(7) $\Rightarrow$ (8). Let n be the number of vertices in G. Assume that the diameter of all obstructions on G is bounded above by a positive integer m, and suppose that there exist an infinite number of obstructions for G. Then there certainly exists an obstruction (H, f) on G where $|H| \geqslant \sum_{j=0}^{m+1} n^i$. We can then use this obstruction (H, f) to construct a sequence of obstructions (H_i, f_i), $1 \leqslant i \leqslant m+1$, where H_i has diameter i, contradicting our original assumption.

To construct the above sequence of obstructions, we need the following claim:

Claim. *Let G a graph with n vertices. Let (H, f) be an obstruction on G and let w be a vertex of H such that*

(i) *$A \cup B \cup C$ is a partition of the vertices of H.*
(ii) *B and C are nonempty.*
(iii) *There are no edges between vertices in A and those in C.*
(iv) *$w \in A \cup B$.*

Then there exists an obstruction (H', f') *and a vertex* w' *of* H' *such that*

(I) $A' \cup B' \cup C'$ *is a partition of the vertices of* H'.
(II) $|B'| \leqslant |B|$, $|C| \leqslant |C'|$ *and* B' *is nonempty.*
(III) *There are no edges between vertices in* A' *and vertices in* C'.
(IV) $w' \in A' \cup B'$.
(V) $d_{H'}(w', B') \geqslant d_H(w, B)$.
(VI) *The number of vertices in* C' *that have a neighbour in* B' *is at most* $n|B|$.

Let H_C denote the subgraph of H induced by the vertices in C. As B is nonempty, there exists an extension t of $f|_{H_C}$ to H_C. For each such t, there exists an obstruction $(Q_t, g_t) \subseteq (H, f \cup t)$. Note that $Q_t \cap C$ is nonempty else $(Q_t, g_t) \subset (H, f)$, contradicting the minimality of (H, f). Moreover, all of $Q_t \cap C$ must be coloured by g_t as all of C is coloured by t.

Let Q be the graph created by taking copies of the graph H_C and all the graphs Q_t as described above, and identifying the vertices in C. We will denote the copy of H_C in Q by $\widehat{H}_C$, and the copy of Q_t in Q by $\widehat{Q}_t$. We give Q the partial colouring g, where g inherits $f|_{H_C}$ on $\widehat{H}_C$ and $f|_{Q_t \setminus C}$ on $\widehat{Q}_t$ for all t. Then (Q, g) is non-extendible, since for any extension t of g to $\widehat{H}_C$, there exists a copy of (Q_t, g_t) contained in (Q, g).

Let $(H', f') \subseteq (Q, g)$ be an obstruction. There exists a homomorphism h from H' to H that maps each vertex x of H' to the vertex of H from which it was copied. The map h must be onto, else (H', f') would be extendible by the minimality of (H, f). Let w' be a vertex of H' such that $h(w') = w$. Let s be the particular extension of $f|_{H_C}$ such that w' is a vertex of $\widehat{Q}_s$. Let A' be $h^{-1}(A) \cap \widehat{Q}_s$, let B' be $h^{-1}(B) \cap \widehat{Q}_s$ and let C' be the rest of H'.

Clearly, $A' \cup B' \cup C'$ is a partition of H'. By construction, $|B'| \leqslant |B|$. Consider $\widehat{H}_C$. As h is onto, $\widehat{H}_C \subseteq H'$. Therefore $|C'| \geqslant |C|$. By construction, $w' \in A' \cup B'$, and also by construction, there are no edges between vertices in A' and those in C'. Thus any path from w' to a vertex in C' (at least one exits as H' is connected) must pass through B', implying that $B' \neq \emptyset$. We have $d_{H'}(w', B') \geqslant d_H(w, B)$ as

$$d_{H'}\big(w', B'\big) \geqslant d_{Q_s}(w, B \cap Q_s) \geqslant d_H(w, B).$$

Now all that is left to prove is (VI). By Lemmas 2.3 and 2.4, a vertex in an obstruction on G has at most n coloured neighbours. If we apply this to (Q_s, g_s), we deduce that all the vertices of $B \cap Q_s$ have at most $n|B \cap Q_s|$ coloured neighbours in Q_s; note that by Lemma 2.4 there are no edges between coloured vertices. As all of $Q_s \cap C$ is coloured in (Q_s, g_s), at most $n|B \cap Q_s|$ vertices of $Q_s \cap C$ have neighbours in $Q_s \cap B$. Therefore at most $n|B|$ vertices of C' have neighbours in B'.

Now we will use the claim to construct (H_i, f_i), $0 \leqslant i \leqslant m+1$, with vertex sets A_i, B_i and C_i and vertex a_i such that

- $A_i \cup B_i \cup C_i$ is a partition of the vertices of H_i.
- $a_i \in A_i \cup B_i$.
- $|B_i| \leqslant n^i$, $B \neq \emptyset$ and $|C_i| \geqslant |H| - \sum_{j=0}^{i} n^i$.
- There are no edges between A_i and C_i.
- $d_{H_i}(a_i, B_i) \geqslant i$.

Let (H, f) an obstruction on G such that $|H| \geqslant \sum_{j=0}^{m+1} n^j$, where m is an upper bound on the diameter of all obstructions on G. We start by setting $H_0 = H$, $A_0 = \emptyset$, $B_0 = \{a_0\}$ and $C_0 = H \setminus a_0$, where a_0 is some vertex of H. Clearly H_0, A_0, B_0, C_0 and a_0 satisfy the statements when $i = 0$.

Suppose that $i \geqslant 1$, and that the statements are true for $i - 1$. Apply the claim to (H_{i-1}, f_{i-1}) with $A = A_{i-1}$, $B = B_{i-1}$, $C = C_{i-1}$ and $w = a_{i-1}$. Let (H_i, f_i) be the resulting obstruction. Let $A_i = A' \cup B'$, let B_i be the vertices in C' that have a neighbour in B', and let $C_i = H_i \setminus (A_i \cup B_i)$. Lastly, let $a_i = w'$.

Clearly $A_i \cup B_i \cup C_i$ is a partition of the vertex set of H_i. It is also easy to see that $a_i \in A_i \cup B_i$ and that $d_{H_i}(a_i, B_i) \geqslant i$. By construction, there are no edges between A_i and C_i. We see that B_i is nonempty as B' and C' are nonempty. In addition, by (VI),

$$|B_i| \leqslant n|B| = n|B_{i-1}| \leqslant n\left(n^{i-1}\right) = n^i.$$

Since $C_i = C' \setminus B_i$ and $B_i \subseteq C'$ and by (II),

$$|C_i| = \left|C'\right| - |B_i| \geqslant |C_{i-1}| - |B_i| \geqslant |H| - \sum_{j=0}^{i-1} n^j - n^i = |H| - \sum_{j=0}^{i} n^j.$$

(8) $\Rightarrow$ (1). Assume that there exists a finite number of obstructions for G. Thus there exists an integer k such that all obstructions on G have at most $k - 1$ coloured vertices. Therefore by Lemma 2.1, G admits a near-unanimity operation of arity k. $\square$

Proof of Theorem 1.1. If G has a near-unanimity operation then its components also have one as they are retracts of G. Indeed, if f is an n-ary near-unanimity operation on G and r is a retraction onto H then the restriction of $r \circ f$ to H^n is a near-unanimity operation on H. Conversely, if each of the components has a near-unanimity operation then the obstructions for G are only the obstructions of the components and the paths with the two endpoints coloured from different components. So the maximum number of coloured elements in an obstruction is finite and G admits a near-unanimity operation.

From this point on we assume that G is connected. Suppose first that G admits a near-unanimity operation. By Theorem 3.1 there is a path $\pi_1 = f_0, f_1, \ldots, f_k = \pi_2$ of idempotents between the two projections in $\mathrm{Hom}(G^2, G)$. As G is reflexive, we may assume that k is even, i.e., $k = 2m$. Note that (f_i, f_j) is a vertex of $\mathrm{Hom}(G^2, G^2)$. Moreover, (f_i, f_j) and (f_{i+1}, f_{j-1}) are adjacent in $\mathrm{Hom}(G^2, G^2)$. Hence

$$(\pi_1, \pi_2) = (f_1, f_{2m}), (f_2, f_{2m-1}), \ldots, (f_m, f_m)$$

is a path in $\mathrm{Hom}(G^2, G^2)$. In fact we claim that the path is from the identity to a retraction onto the diagonal. Clearly $(\pi_1, \pi_2)(x, y) = (x, y)$. The range of (f_m, f_m) is the diagonal of G^2 and $(f_m, f_m)(x, x) = (x, x)$ as f_m is idempotent. Moreover, each of the maps in the path fixes the diagonal. Hence by Lemma 2.5, G^2 dismantles to the diagonal. Conversely, assume that there exists a path in $\mathrm{Hom}(G^2, G^2)$ from the identity map to a retraction onto the diagonal such that the diagonal is contained in the image of each of these maps. Let $h_0, h_1, \ldots, h_m$ be such a path. Then $\pi_1 \circ h_0, \pi_1 \circ h_1, \ldots, \pi_1 \circ h_m = \pi_2 \circ h_m, \ldots, \pi_2 \circ h_1, \pi_2 \circ h_0$ is the desired path in $\mathrm{Hom}(G^2, G)$. $\square$

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