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Idempotent Totally Symmetric Operations on Finite Posets

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Abstract. An n-ary operation f is totally symmetric if it obeys the identity $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$ for all sets of variables such that $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$. We characterize finite posets admitting an n-ary idempotent totally symmetric operation for all n. The characterization is expressed in terms of zigzags, special objects related to the poset. Some open problems concerning idempotent Malcev conditions for order primal algebras are mentioned.

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1. Introduction

An algebra is called order primal if its term operations coincide with the monotone operations of a poset. Order primal algebras might serve as test algebras to answer questions for a broader spectrum of algebras. In papers [1–3, 8, 9] and [11] the interested reader can find various algebraic properties which are investigated for order primal algebras. The study of idempotent identities satisfied in order primal algebras were always in the center of these investigations. Usually, nontrivial idempotent Malcev conditions come into play. For the definition of a nontrivial idempotent Malcev condition see for, e.g., [5, Chapter 9].

We say that a poset P admits operation f if the order relation of P is preserved by f, i.e., f is monotone with respect to P. A poset P admits a Malcev condition if the Malcev condition holds in an order primal algebra related to P. This is equivalent to saying that P admits monotone operations satisfying the appropriate identities prescribed in the definition of the given Malcev condition.

When one tries to solve problems of algebraic nature on order primal algebras one frequently encounters problems such as for a given nontrivial idempotent Malcev condition to describe the order structure of the posets admitting it. It would be useful in the investigations if we knew the order structure of the posets that admit any kind of nontrivial idempotent Malcev conditions. These are the posets that admit a so called Taylor operation, cf. [10].

Operations satisfying the identity $f(x, x, \dots, x) = x$ are called idempotent. Let $n \geq 2$. An n-ary Taylor operation is an n-ary idempotent operation f satisfying n identities of the form

$$f(\dots, \underbrace{x}_{i}, \dots) = f(\dots, \underbrace{y}_{i}, \dots), \quad i \in \{1, \dots, n\}$$

where x and y are the only variables occurring in the identities and $x \neq y$. There are examples of posets with and without monotone Taylor operations. Some fairly nice classes of posets with nontrivial Malcev conditions were described in [8, 9] and [11]. On the other hand, in [7] Larose proved that sums of nontrivial ramified posets over a nontrivial connected poset admit only the trivial idempotent operations, namely the projections. Crowns are also known to have this property, see [3]. Since Taylor operations are idempotent and definitely different from projections these posets do not admit a Taylor operation.

To describe the order structure of posets admitting a Taylor operation does not seem to be an easy matter. Next we shall study special Taylor operations and try to characterize the order structure of posets admitting them.

Let $n \geq 2$. An n-ary operation f is called symmetric if it obeys the identity

$$f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$$

for every permutation π of $\{1, \dots, n\}$. The operation f is called totally symmetric if it obeys the identity

$$f(x_1, \dots, x_n) = f(y_1, \dots, y_n)$$

for all sets of variables such that $\{x_1, \dots, x_n\} = \{y_1, \dots, y_n\}$. Interestingly enough, all known examples of finite posets which admit a nontrivial idempotent Malcev condition admit an n-ary idempotent totally symmetric operation for every n. They even have a compatible semilattice operation. Some of them are depicted in Figure 1. This suggests that the existence of idempotent totally symmetric term

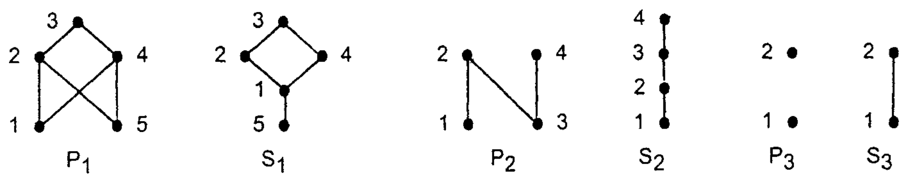

Figure 1. Posets admitting a semilattice operation and related join-semilattices.

operations might play a crucial role when studying nontrivial idempotent Malcev conditions in order primal algebras.

In the paper we give a necessary and sufficient condition for a poset to admit an n-ary idempotent totally symmetric operation for every n. The condition given makes use of certain objects called zigzags that are related to the poset. Zigzags have already played an important role in the characterizations of order primal algebras with particular nontrivial idempotent Malcev conditions such as the Malcev condition for congruence distributivity or modularity of the variety generated by the algebra and the existence of a near unanimity term operation, see [8] and [11]. Let $n \geq 3$. An n-ary operation f is called a near unanimity operation if the following n identities hold:

$$f(x, \ldots, x, \underbrace{y}_{i}, x, \ldots, x) = x, \quad i \in \{1, \ldots, n\}.$$

2. Results

It is an easy exercise that the commutative idempotent groupoid freely generated by three elements does not have a ternary symmetric term operation. So possessing an n-ary idempotent (totally) symmetric term operation for some n is not a Malcev condition although (totally) symmetric idempotent term operations are Taylor operations. In case of order primal algebras the situation is somewhat nicer. It follows from the proposition below that if a finite poset P admits an n-ary idempotent totally symmetric operation for $n \geq |P|$, then it admits a k-ary one for every k.

PROPOSITION 1. (1) *Let A be any algebra. If n is at least 3 and there exists an n-ary idempotent totally symmetric term operation of A then there also exists an $(n-1)$-ary operation.*

(2) *Let A be an order primal algebra corresponding to order $\leq$. If n is at least $|A|$ and f is an n-ary idempotent totally symmetric term operation of A then f satisfies the property that $f(a_1, \ldots, a_n) \leq f(b_1, \ldots, b_n)$ whenever for each a_i there exists some b_j with $a_i \leq b_j$ and, dually, for each b_i there exists some a_j with $a_j \leq b_i$.*

(3) *Let A be an order primal algebra. If n is at least $|A|$ and there exists an n-ary idempotent totally symmetric term operation of A, then there also exists an $(n+1)$-ary operation.*

Proof. Let f be an n-ary idempotent totally symmetric term operation of A. Obviously, by identifying two variables of f we get an $(n-1)$-ary idempotent totally symmetric term operation of A. So (1) holds. Suppose that A is order primal with respect to the order $\leq$ and n is at least $|A|$. Now, by using that f is monotone with respect to $\leq$ and totally symmetric, (2) follows from the fact that any n-tuple in A^n has at most $|A|$ different components. The condition that n is at least $|A|$ is needed to guarantee that the components of two n-tuples in question can be arranged appropriately. In order to prove (3) we define an $(n+1)$-ary operation g

as follows. Let $g(b_1, \dots, b_{n+1}) = f(a_1, \dots, a_n)$ for each $(b_1, \dots, b_{n+1}) \in A^{n+1}$ where $\{a_1, \dots, a_n\} = \{b_1, \dots, b_{n+1}\}$. Operation g is well defined and totally symmetric since f is totally symmetric. Moreover, g is monotone since (2) holds for f. The idempotency of g is clear. □

A full tree of depth l is a rooted tree whose leaves are at distance l from the root. Let $i \le l$. A full subtree of depth i is the subtree spanned by the vertices of distance at most i from the root in a full tree of depth l. Let C and S be arbitrary sets. A (partial) map $f : S \to C$ is called a coloring of S by C. Then we call S a C-colored or $|C|$-colored set. If $f(s)$ is defined for some $s \in S$ then we say that s is a colored element and $f(s)$ is the color of s.

LEMMA 2. *Let c and l be arbitrary positive integers with $c < l$. Then any c-colored full tree of depth l whose coloring is fully defined contains a full subtree of depth at least $l - c$ such that for each leaf of the subtree there is a non-leaf vertex of the same color.*

Proof. Let P_l be a c-colored full tree of depth l whose coloring is fully defined. For each i with $l - c \le i \le l$ let C_i denote the set of colors of the elements of distance i from the root in P_l. Since there is a c-element set containing all the C_i there exists some i with $l - c < i \le l$ such that $C_i \subseteq \bigcup_{j=l-c}^{i-1} C_j$. So the full subtree formed by the elements of distance at most i from the root in P_l satisfies the claim. □

Let P be a poset. We call a pair (H, f) a P-colored poset if H is a poset and f is a partial map from H to P. We say that (H, f) is extendible if f extends to a fully defined monotone map from H to P. For arbitrary posets K and H we write $K \subseteq H$ if the base set and the order relation of K are contained in the base set and in the order relation of H, respectively. We call a finite nonextendible P-colored poset (H, f) a P-zigzag if for any poset K with $K \subset H$ the P-colored poset $(K, f|_K)$ is extendible. A poset is called a tree if its covering graph is a tree. A P-colored poset is a tree if its base poset is a tree. Let (H, f) and (G, g) be arbitrary P-colored posets. We say that α is a homomorphism from (H, f) to (G, g) if α is a monotone map from H to G such that $f = g\alpha$. If α is onto we call (G, g) the homomorphic image of (H, f). We say that (H, f) contains (G, g) whenever $G \subseteq H$ and $g \subseteq f$.

Observe that every finite nonextendible P-colored poset contains a P-zigzag. The importance of zigzags is understandable when we are given a finite poset P whose zigzags are described completely and we try to decide whether a certain finite P-colored poset (H, f) is extendible. This is a quite common task when one would like to know that P admits an operation of special kind. In order to prove that (H, f) is extendible it is sufficient to show that there are no zigzags contained in (H, f). When P-zigzags are at hand the latter is easy.

Congruence distributivity and modularity of a variety are each well known to be characterized by the existence of a Malcev condition satisfied by the variety, see

Jónsson [6], Day [2] and also Gumm [4]. The terms occurring in the definitions of these Malcev conditions are commonly called as Jónsson, Day and Gumm terms, respectively. In [8] it is proved that the following conditions are equivalent for any finite connected poset P:

(i) P admits Jónsson operations.
(ii) P admits Gumm operations.
(iii) P admits a near unanimity operation.
(iv) There are only finitely many P-zigzags up to isomorphism.
(v) The poset of binary idempotent monotone operations on P is connected.

This is an example how information on zigzags yields information on the existence of a Malcev condition in an order primal algebra. Our next theorems are formulated in a similar fashion.

THEOREM 3. *A finite poset P admits an n-ary idempotent totally symmetric operation for every n if and only if every P-zigzag is a homomorphic image of a nonextendible tree.*

Proof. Suppose first that P admits an n-ary idempotent totally symmetric operation for every n. Let (H, f) be an arbitrary P-zigzag. We shall construct a nonextendible tree (G_l, g_l) such that (H, f) is a homomorphic image of (G_l, g_l).

Let us fix a positive integer $l \geq (|P|+1)|H|$ and an element h_0 of H. We define a P-colored tree (G_l, g_l) as follows. Let the base set of G_l be

$$\{(a_0, \ldots, a_k) : a_0 = h_0, a_1, \ldots, a_k \in H, a_i \prec a_{i+1} \text{ or } a_{i+1} \prec a_i,$$
$$0 \leq i < k, k \leq l\}.$$

We define the covering pairs in G_l by $s \prec s'$ $(s' \prec s)$ if

$$s = (a_0, a_1, \ldots, a_k), \qquad s' = (a_0, a_1, \ldots, a_k, a_{k+1})$$

and

$$a_k \prec a_{k+1} \; (a_{k+1} \prec a_k).$$

So the base set of G_l consists of the paths in the covering graph of H that start from h_0 and are of length at most l and two paths form a covering pair in G_l if one is the continuation of the other by one element. Roughly speaking, the covering graph of G_l is the union of all paths of the covering graph of H that start from h_0 and are of length at most l with two paths identified until they part. Clearly, G_l is a tree. Let (h_0) be assigned as the root of G_l. Then G_l is a full tree of depth l.

Let $g_l(s) = f(a_k)$ whenever $s = (a_0, \ldots, a_k) \in G_l$ and $f(a_k)$ is defined. Now, observe that (H, f) is a homomorphic image of (G_l, g_l) under the homomorphism $r : G_l \to H, (a_0, \ldots, a_k) \mapsto a_k$ because $l \geq |H|$. So it remains to prove that the P-colored tree (G_l, g_l) is nonexendible.

By way of contradiction, let us assume that (G_l, g_l) is extendible and let us take $q : G_l \to P$ to be an extension of g_l. Then (q, r) is a fully defined $|P||H|$-coloring of the full tree G_l of depth l. Hence Lemma 2 applies. So G_l contains a full subtree H' of depth at least $l - |P||H| \geq |H|$ such that in H' for each leaf there is a non-leaf vertex of the same color. Let $f' = g_l|_{H'}$.

The map $v = r|_{H'}$ is a homomorphism from (H', f') to (H, f). So $fv = f'$. Moreover, v is onto since the depth of H' is at least $|H|$.

Let β denote a totally symmetric operation on P of arity $|P|$. For any $h \in H$ let t_h be any $|P|$-tuple whose set of components is $\{q(s) : v(s) = h, s \in H'\}$. Note that the latter set is nonempty, since v is onto. We define a map $\alpha : H \to P$ by setting $\alpha(h) = \beta(t_h)$. Now, α is well defined since β is totally symmetric. The map α extends f since q extends g_l and β is idempotent.

In order to prove that α is monotone let $h \prec h'$ in H. By part (2) of Proposition 1, $\alpha(h) \leq \alpha(h')$ will follow if we show that for every $s \in H'$ where $v(s) = h$ there is an $s' \in H'$ such that $v(s') = h'$ and $q(s) \leq q(s')$ (and that the dual claim also holds). This is obvious if s is a non-leaf element of H'. If s is a leaf of H' then by the definition of H' there is a non-leaf vertex s'' of H' such that $v(s'') = v(s) = h$ and $q(s'') = q(s)$. Hence we are done in that case, too.

So $\alpha : H \to P$ is a monotone map that extends f. This contradicts the fact that (H, f) is nonextendible. Hence, (G_l, g_l) is nonextendible. This concludes the proof of one direction.

To prove the other direction suppose that P is a finite poset such that every P-zigzag is a homomorphic image of a nonextendible tree. Let n be an arbitrary integer at least 2. We define an equivalence relation θ on P^n by $(a_1, \ldots, a_n)\theta(b_1, \ldots, b_n)$ if and only if the sets of extremal elements of subposets spanned by $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in P are equal. Now we define a relation on the set of blocks of θ by $A \leq B$ if and only if whenever $a \in A$ and $b \in B$ for each component a_i of a there exists a component b_j of b with $a_i \leq b_j$ and, dually, for each component b_i of b there exists a component a_j of a with $a_j \leq b_i$. Observe that the set of blocks of θ with the relation defined above is a poset. We let P^n/θ denote this poset.

We shall define a partial map g from P^n/θ to P and show that the P-colored poset $(P^n/\theta, g)$ is extendible. This will enable us to define an n-ary idempotent totally symmetric operation on P.

Let g be defined by $g(A) = p$ for every one element block $A = \{(p, p, \ldots, p)\}$ of P^n/θ. Suppose by a contradiction that $(P^n/\theta, g)$ is nonextendible. Then it contains a zigzag which by the assumption is a homomorphic image of a nonextendible tree, say (H, f). So there is a monotone map $s : H \to P^n/\theta$ such that $f = gs$.

We define a monotone map $s' : H \to P$ such that $s'(h)$ is one of the components of some element in $s(h)$ for every $h \in H$. First, we take a list $h_1, \ldots, h_m$ of the elements of H in such a way that the consecutive elements on the list are connected via an edge in the covering graph of H. Now, s' is defined recursively on the elements of H as follows. Choose $s'(h_1)$ to be any component of an arbitrary element of $s(h)$. Suppose that we have defined s' for $h_1, \ldots, h_n$ where $1 \leq n < m$ such that $s'(h_i)$

is one of the components of an element in $s(h)$ for each $1 \leq i \leq n$. Without loss of generality we assume that $h_n \leq h_{n+1}$. Hence $s(h_n) \leq s(h_{n+1})$. Since $s'(h_n)$ equals a component of some element of $s(h_n)$ there is an element in $s(h_{n+1})$ such that for its j-th component b_j we have that $s'(h_n) \leq b_j$. Let $s'(h_{n+1}) = b_j$. Now, s' is defined on the entire H and since H is a tree, s' is monotone. When $f(h)$ is defined $s(h)$ contains a single element whose components coincide with $f(h) = g(s(h))$. As $s'(h)$ is one of the components of some element in $s(h)$, s' extends f. Thus, (H, f) is extendible, a contradiction.

So the colored poset $(P^n/\theta, g)$ does not contain any P-zigzag. Hence it is extendible. Let g' be an extension of g to P^n/θ. By composing the natural map corresponding to θ with g' we obtain a monotone n-ary idempotent totally symmetric operation on P. □

Due to the fact that every finite poset is a homomorphic image of a tree, every zigzag is a homomorphic image of a tree that might be either extendible or nonextendible. Now, in view of the examples mentioned in the Introduction, Theorem 3 just implies that if every P-zigzag is a homomorphic image of a nonextendible tree then P must be quite special. In the statement of Theorem 3 we would have liked to replace 'nonextendible tree' by 'tree zigzag'. But we were not able to give a proof in that case. Next we shall see that under special conditions on P this replacement is possible.

A zigzag (H, f) is maximal provided that every zigzag (G, g) from which there is homomorphism to (H, f) is isomorphic to (H, f).

THEOREM 4. *Let P be a finite poset. If P admits an n-ary idempotent* (*totally*) *symmetric operation for every n then the maximal P-zigzags are trees.*

Proof. Let P be a finite poset such that P admits an n-ary idempotent symmetric operation for every n. Let (H, f) be a maximal P-zigzag. Suppose that (H, f) is not a tree zigzag.

We shall define a colored poset for every integer k. These colored posets will turn out to be extendible. This and the existence of a monotone n-ary idempotent symmetric operation for an appropriate n will lead to the contradiction that (H, f) is extendible.

Let a and b be two elements in H such that b covers a, moreover (a, b) lies in a circuit in the covering graph of (H, f). Let k be any integer at least 2. We define the colored poset (G_k, g_k) as follows. We take a copy of (H, f) for each $i = 1, \ldots, k$ and denote it by (H_i, f_i). The elements of (H_i, f_i) are referred according to the elements of (H, f) and indexed by i. So each (H_i, f_i) has a covering pair of the form (a_i, b_i). Now, (G_k, g_k) is obtained by deleting (a_i, b_i) in (H_i, f_i) for all i and connecting the remaining k colored posets by new covering edges (a_i, b_{i+1}) for each $i = 1, \ldots, k-1$. We claim that (G_k, g_k) is extendible. If not then (G_k, g_k) contains a zigzag (G, g).

Observe that there exists a j such that G contains the edge (a_j, b_{j+1}). This follows from the fact that $(H, f) - \{(a, b)\}$ is extendible and is a homomorphic image of $(G_k, g_k) - \{(a_1, b_2), \ldots, (a_{k-1}, b_k)\}$ under the homomorphism $h_i \mapsto h$. So the latter colored poset is also extendible. It follows similarly that for any covering pair (c, d) different from (a, b) in H there is an i such that G contains (c_i, d_i).

Let v denote the map $G \to H$ where $h_i \mapsto h$. Clearly, v is a homomorphism from (G, g) to (H, f). Let $x^0 = a, x^1, \ldots, x^n = b, x^0 = a$ be a circuit in the covering graph of (H, f). Let us take a j such that $(a_j, b_{j+1}) \in G$. Notice that G contains $a_j = x_j^1, \ldots, x_j^n = b_j$. For otherwise, by taking the above observations into account there are two different elements of G, namely x_j^l and $x_{j'}^l$ for some l and $j' \neq j$, mapped into the same value by v, which contradicts to the maximality of (H, f). So G contains b_j and b_{j+1}. But then these two elements are mapped into the same element by v, a contradiction. So (G_k, g_k) is extendible for any integer k.

Let $g_k' : G_k \to P$ be an extension of g_k. If k is large enough there exist l and m with $l + 2 \leq m \leq k$ such that $g_k' \mid_{H_l} = g_k' \mid_{H_m}$. Let β be an $(m - l)$-ary idempotent totally symmetric operation admitted by P. We define the map

$$\alpha : H \to P \quad \text{where } h \mapsto \beta(g_k'(h_l), \ldots, g_k'(h_{m-1})).$$

The map α extends f since $g_k' \mid_{H_i}$ extends f_i for all $i \leq k$ and β is idempotent. On the other hand α is monotone. The only covering pair for which the monotonicity of α is not clear is (a, b). In this case the symmetricity of β ensures that $\alpha(a) \leq \alpha(b)$. Thus, (H, f) is extendible. This contradiction concludes the proof of Theorem 4. □

We note that for the totally symmetric part of Theorem 4 a short proof could be given by using the (G_l, g_l) construction from the proof of Theorem 3. The following corollary is a nicer version of Theorem 3 in the special case when poset P is connected and satisfies any of the equivalent conditions (i)–(v).

COROLLARY 5. *Let P be a finite poset with finitely many zigzags. Then P admits an n-ary idempotent* (*totally*) *symmetric operation for every n if and only if every P-zigzag is a homomorphic image of a tree zigzag.*

Proof. Suppose that P is a finite poset with finitely many zigzags. Then every P-zigzag is a homomorphic image of a maximal P-zigzag. Hence by Theorem 3 and 4 we get the claim. □

3. Concluding Remarks

It is easy to check that for any finite poset P each of the following conditions implies the ones below it.

P admits a semilattice operation.
P admits an n-ary idempotent totally symmetric operation for every n.
P admits an n-ary idempotent symmetric operation for every n.
P admits a Taylor operation.

At present we are not able to prove that the classes of finite posets determined by these four conditions are different. We would like to see either examples of posets showing that the above four conditions are pairwise nonequivalent, or proofs showing the equivalence of some of them. It is also not known whether there exists a finite poset that admits a near unanimity operation but no idempotent totally symmetric operations.

In Theorem 3 we characterized finite posets that admit an n-ary idempotent totally symmetric operation for every n in terms of their zigzags. Is there a description of these posets similar to the one given in (v) for finite connected posets admitting a near unanimity operation?

In [5] Hobby and McKenzie define the type set of a variety. In this respect, we note that all varieties generated by an order primal algebra related to a finite poset which satisfies the equivalent conditions of Theorem 3 omit types 1 and 2. We do not know whether they also omit type 5.

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