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Characterizing finite irreducible relational sets

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Abstract. We associate a certain lattice with every relational set. We characterize finite irreducible relational sets by the property that their associated lattice leaves a lattice if its top element is removed. This characterization is somewhat dual to that of subdirectly irreducible algebras by their congruence lattices. As a corollary we prove that if the idempotent clone related to a finite relational set P is trivial then P is irreducible. A stronger version of irreducibility is also explored.

1. Introduction

A *relational set* for us is a set equipped with some (possibly infinitary) relations. Relational sets here are assumed to have some definite type. *Morphisms* between relational sets of the same type are relation preserving maps. *Retract* and *product* in the class of relational sets of the same type are meant as usual in category theory. A relational set is *finite* if its base set is finite and its relations are finitary.

In their seminal 1981 paper [4] Duffus and Rival introduced the notions: representation of a partially ordered set, irreducible poset. These concepts carry over to arbitrary relational sets without any change. So we call a sequence $P_i, i \in I$ of relational sets a *representation of a relational set* P if P_i and P are of the same type for all $i \in I$, P is a retract of $\prod_{i\in I} P_i$ and the P_i are retracts of P for all $i \in I$. A representation $P_i, i \in I$ is *finite* if I is finite and the relational sets P_i are finite for all $i \in I$. A relational set P is *irreducible* if for any representation $P_i, i \in I$ of P there exists an i such that P is a retract of P_i.

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In their paper [4] Duffus and Rival gave various reasonings why they chose the above definitions of representation and irreducibility for posets. Their main motivation was Birkhoff's subdirect representation theorem for algebras. Unfortunately, it came in [4] as an open problem whether every poset has a representation via irreducible posets. This problem has been open since then. Nevertheless, it is easy to prove a Birkhoff type theorem for finite posets. Duffus and Rival showed in [4] that every finite poset has a finite representation via irreducible posets. It is also easy to get the same result for finite relational sets, see [8].

A key feature corresponding to Birkhoff's theorem is the characterization of subdirectly irreducible algebras as the ones whose congruence lattice has a smallest element among its nonzero elements. Interestingly enough an analogue of this characterization could be obtained for finite irreducible relational sets. In Section 2 we state and prove this.

Let D be a set. A unary operation r on D is called *idempotent* if $r^2 = r$. The main idea in the proof of the characterization theorem is that we conceive a representation of P as a certain sequence of idempotent endomorphisms of P rather than the one of their images. This categorical point of view makes it possible to manipulate with compositions of morphisms.

Let P be a relational set in a class K of relational sets of the same type as P. We call P *strongly irreducible* in K if for any relational sets Q and R in K if P is a retract of $Q \times R$ then P is a retract of Q or R. Clearly, any strongly irreducible relational set P in K is irreducible whenever K contains all retracts of P. Strong irreducibility relativised to posets has been studied in [2],[6] and implicitly present in [4] and [7]. On strong irreducibility for other binary structures see [5]. In Section 3 we prove a result related to strongly irreducible relational sets and provide examples of irreducible relational sets that are not strongly irreducible in familiar classes of finite binary relational sets.

2. A characterization of irreducible relational sets

The new characterization of irreducible relational sets in this section is related to two earlier results on categorical equivalence of algebras. In order to state these results we need to introduce some definitions.

We say that algebra A is *categorically equivalent* to algebra B if there exists a categorical equivalence between the varieties they generate such that $F(A) = B$. An algebra A and B are called *weakly isomorphic* if there is an algebra C such that C and A are of the same type, C is isomorphic to A, the base set of B equals the one of C and the term operations of B and C coincide. For an algebra B and an idempotent unary term operation r on B let $r(B)$ denote the algebra whose base set is the image of r and whose basic operations are obtained by restricting the restrictible term operations of B to the image of r. An algebra A is called *c-minimal* if for every algebra B categorically equivalent to A there is an idempotent unary term operation r of B such that A is weakly isomorphic to $r(B)$. For an algebra A let $\mathrm{Sub}(A)$ denote the lattice of subuniverses of A.

One of the above mentioned results was proved by Bergman and Berman in [1]. It states that if A is a finite algebra, F_1 is the free algebra freely generated by one element in the variety generated by A and $\mathrm{Sub}(F_1) \setminus \{F_1\}$ has a largest element then A is c-minimal.

Let A be an algebra and let P be a relational set. We say that P is a *relational set for* A if the base sets of P and A are the same and the set of finitary term operations of A coincides with the set of morphisms from finite powers of P to P. A *relation variety* is a class of relational sets closed under product and retract.

Let A be a finite algebra and let P be a finite relational set for A. Suppose that P is irreducible. By Theorem 1.14 in [8], P is a retract of every finite relational set Q whenever P and Q generate the same relation variety. By Theorem 2.3 in [8] this is equivalent to saying that A is c-minimal. Then the second result is as follows. If A is a finite algebra and P is a finite irreducible relational set for A then A is c-minimal.

So we have two results with the same conclusion and with seemingly different premises. The main theorem in this chapter states that these premises, in fact, are equivalent. The following lemma is needed in the sequel.

Lemma 2.1. *Let P be a finite relational set. Let H be the set of non-onto endomorphisms of P. Then there exists a positive integer m such that for any $t_1, \dots, t_m$ in H we have that $t_m \dots t_1 = srq$ for some s, q and idempotent r in H.*

Proof. Let $m = 2^{|P|} + 1$ Then there are indices i and j such that $i < j$ and $t_i \dots t_1(P) = t_j \dots t_1(P)$. By the finiteness of P it follows that $t_j \dots t_{i+1}$ is bijective on $t_i \dots t_1(P)$. Let r be an idempotent power of $t_j \dots t_{i+1}$. Then r is the identity restricted to $t_i \dots t_1(P)$. Hence, $t_m \dots t_1 = t_m \dots t_{i+1} r t_i \dots t_1$. So, we have the claim. ■

Let D be a set. An n-ary operation f on D is called idempotent if it obeys the identity $f(x, x, \dots, x) = x$. This definition interferes with the notion of idempotent unary operation defined earlier. Later on it will always be clear from the context which notion of idempotency is used.

Theorem 2.2. *Let A be a finite algebra and let P be a finite relational set for A. Let F_1 be the subalgebra generated by id_A in algebra A^A. Then the following are equivalent.*

(1) *P is irreducible.*

(2) *The non-onto unary term operations of A, i.e., the non-onto endomorphisms of P form a subuniverse of F_1.*

(3) *For all n-ary idempotent term operation h, n-ary term operation t and unary term operations $t_1, \dots, t_n$ of A if $h(x_1, \dots, x_n) = t(t_1(x_1), \dots, t_n(x_n))$ then there exists an i such that t_i is onto.*

(4) $\mathrm{Sub}(F_1) \setminus \{F_1\}$ *has a largest element.*

Proof. Let H be the set of non-onto morphisms in F_1. First we show (1) implies (2). Suppose (2) does not hold. Then for algebra A there exists an n-ary term operation u_1 and unary non-onto term operations $t_1 \dots t_n$ such that $u_1(t_1, \dots, t_n) = \mathrm{id}_A$. Then $u_1(t_1 t_i, \dots, t_n t_i) = t_i$ for all $i = 1, \dots, n$. Hence

$$u_1(u_1(t_1 t_1, \dots, t_n t_1), \dots, u_1(t_1 t_n, \dots t_n t_n)) = u_1(t_1, \dots, t_n) = \mathrm{id}_A \,.$$

So there is an n^2-ary term operation u_2 of A such that

$$u_2(t_1 t_1, \dots, t_n t_1, \dots, t_1 t_n, \dots, t_n t_n) = \mathrm{id}_A \,.$$

By proceeding in this fashion we get the existence of an n^m-ary term operation u_m of A such that

$$(*) \qquad u_m((t_{i_1} \dots t_{i_m})_{1 \le i_1, \dots, i_m \le n}) = \mathrm{id}_A$$

where m is an integer whose existence is guaranteed by Lemma 2.1. Let $L = \{(i_1, \dots, i_m) : \ 1 \le i_1 \dots i_m \le n\}$. Let $l = (i_1, \dots, i_m)$ be an arbitrary element of L. By Lemma 2.1, we have that $t_{i_1} \dots t_{i_m} = s_l r_l q_l$ for some s_l, q_l and idempotent r_l in H. Observe now that the morphism $P \to \prod_{l \in L} r_l(P)$, $a \mapsto (r_l g_l(a))_{l \in L}$ is a coretraction by $(*)$. Since r_l is non-onto $r_l(P)$ is a proper retract of P for each $l \in L$. Hence P is reducible, i.e., (1) does not hold.

Suppose that (2) holds. Let us take an n-ary idempotent term operation h, an n-ary term operation t and unary term operations $t_1, \dots, t_n$ of A with the property that $h(x_1, \dots, x_n) = t(t_1(x_1), \dots, t_n(x_n))$. Since h is idempotent $t(t_1, \dots, t_n) = \mathrm{id}_A$. Since H is a subalgebra of F_1 and $\mathrm{id}_A \notin H$ the preceding equality implies that there exists an i such that t_i is onto. So (3) holds.

If (3) holds then the largest element of $\mathrm{Sub}(F_1) \setminus \{F_1\}$ is formed by the non-onto unary term operations of A. For otherwise there exist an n-ary term operation

t and non-onto unary term operations $t_1, \ldots, t_n$ of A satisfying $\mathrm{id}_A = t(t_1, \ldots, t_n)$. Let $h(x_1, \ldots, x_n) = t(t_1(x_1), \ldots, t_n(x_n))$. Then h is an idempotent term operation of A contradicting (3). Thus, (4) holds.

Finally, we prove (4) implies (1). Assume that (1) does not hold, i.e., P is not irreducible. Then there exist idempotent $r_1, \ldots, r_n$ in H, a retraction $r: \prod_{i=1}^{n} r_i(P) \to P$ and a corresponding coretraction $e: P \to \prod_{i=1}^{n} r_i(P)$. So $re = \mathrm{id}_A$, i.e.,

$$r(r_1\pi_1 e, \ldots, r_n\pi_n e) = \mathrm{id}_A \quad ,$$

where π_i is the restriction of the n-ary projection in the i-th variable on P to $\prod_{i=1}^{n} r_i(P)$ for each $i = 1, \ldots, n$. This and the facts that $r(r_1(x_1), \ldots r_n(x_n))$ is an n-ary term operation on A and id_A generates F_1 yield $[\pi_1 e, \ldots, \pi_n e] = F_1$. Because the morphisms in $[\pi_1 e], \ldots, [\pi_n e]$ are non-onto each of $[\pi_1 e], \ldots, [\pi_n e]$ is a proper subalgebra of F_1. Since the join of these subalgebras is F_1 condition (4) does not hold.

■

We note that a shorter proof could be given for (1) implies (2) by using a characterization of irreducible relational sets via obstructions in [8].

Corollary 2.3. *Let P be a finite relational set. If all idempotent morphisms from finite powers of P to P are projections then P is irreducible.*

Proof. The corollary immediately follows by condition (3) of Theorem 2.2. ■

3. A stronger notion of irreducibility

Corominas proved in [2] that posets with no nontrivial idempotent monotone operations are strongly irreducible in the class of posets. His proof extends to certain relational sets as follows.

Theorem 3.1. *Let P, Q and R be relational sets of the same type such that there exist morphisms from Q and R to P, moreover P is a retract of $Q \times R$. If the only idempotent morphisms from P^2 to P are projections then P is a retract of Q or R.*

Proof. Let P be a relational set such that the only idempotent morphisms from P^2 to P are projections. Suppose that h_1 is a morphism from P to Q and h_2 is a morphism from P to R. We shall establish the following property for P: for all relational sets Q and R, morphisms $t_1: P \to Q$, $t_2: P \to R$ and $t: Q \times R \to P$ which satisfy the identity $t(t_1(x), t_2(x)) = x$ either t_1 or t_2 is a coretraction. From this the statement of the theorem follows since if P is a retract of $Q \times R$ then there exist t, t_1 and t_2 satisfying the given identity.

So let Q and R be relational sets and let $t_1: P \to Q$, $t_2: P \to R$ and $t: Q \times R \to P$ be morphisms which satisfy the identity $t(t_1(x), t_2(x)) = x$. Then $t(t_1(x), t_2(y))$ is an idempotent morphism from P^2 to P. So without loss of generality $t(t_1(x), t_2(y)) = x$. In which case t_1 is a coretraction from P to Q with $r: Q \to P$, $z \mapsto t(z, t_2 h_1(z))$ as an appropriate retraction. ■

Corollary 3.2. *Let K be a class of relational sets such that from any element of K there is a morphism to any element of K. Let P be in K. If all idempotent morphisms from finite powers of P to P are projections then P is strongly irreducible in K.*

Note that the above mentioned theorem of Corominas is a special case of Corollary 3.2. One can find examples of finite posets admitting no nontrivial idempotent operations in [2], [3] and [6]. In [3] Demetrovics and Rónyai proved that crowns admit no nontrivial idempotent operations. Their result extends to superpositions of crowns and truncated Boolean lattices as in Corominas [2]. In [6] Larose proved that sums of nontrivial ramified posets over a nontrivial connected poset admit no nontrivial idempotent operations. Besides, he presented other examples of posets with the same property.

It has long been an open problem, see [4], whether the notions of irreducibility and strong irreducibility coincide in the class of finite posets. On the other hand, it is easy to come up with an example of an irreducible but not strongly irreducible relational set among finite irreflexive digraphs and is not too hard to find one among finite reflexive digraphs or among finite graphs (symmetric digraphs), see Figure 1.

In the figure shaded nodes symbolize the vertices with loops and undirected edges do back and forth directed edges. All digraphs shown in the figure are irreducible. On each line the first relational set is a retract of the product of the last two but it is not a retract of either of them.

The relational sets are irreflexive digraphs on the first, reflexive digraphs on the second and graphs on the third line. The second example was presented by Kabil and Pouzet in [5].

Note that the first graph in the third example admits only essentially unary operations. Hence this example also demonstrates that in Theorem 3.1 the condition that there are morphisms from Q and R to P is essential.

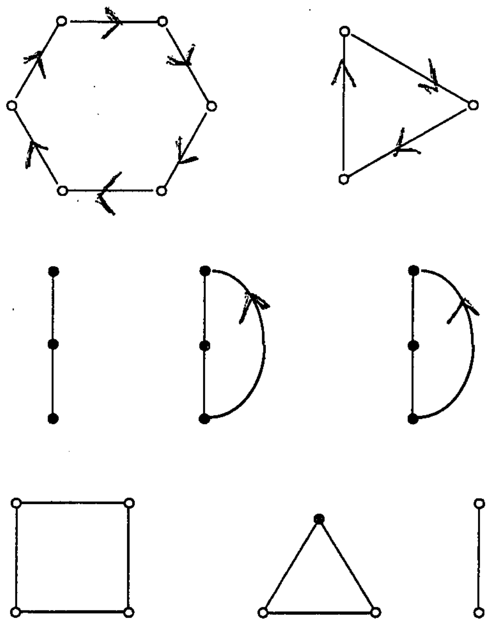

Figure 1.

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