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Finite posets and topological spaces in locally finite varieties

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In Celebration of the Sixtieth Birthday of Ralph N. McKenzie

Abstract. We prove that if a finite connected poset admits an order-preserving Taylor operation, then all of its homotopy groups are trivial. We use this to give new characterisations of locally finite varieties omitting type 1 in terms of the posets (or equivalently, finite topological spaces) in the variety. Similar variants of other omitting-type theorems are presented. We give several examples of posets that admit various types of Taylor operations; in particular, we exhibit a topological space which is not an H-space but is compatible with a set of non-trivial identities, answering a question of W. Taylor.

1. Introduction

Fix a finite relational structure $\mathcal{T}$ of finite type and consider the following decision problem: *given a structure $\mathcal{S}$ of the same type as $\mathcal{T}$, is there a homomorphism from $\mathcal{S}$ to $\mathcal{T}$?* There has been a great deal of interest recently in these so-called *restricted constraint satisfaction problems* (CSP for short), see for example [2, 3, 4, 5, 9, 14, 20, 23]. The main questions concerning the algorithmic complexity of these problems were first formulated by T. Feder and M. Vardi in [14]: (i) (Dichotomy problem) under the assumption that $\mathcal{P} \neq \mathcal{NP}$, is there a structure $\mathcal{T}$ whose decision problem is neither in $\mathcal{P}$ nor $\mathcal{NP}$-complete ? (ii) Classify the structures $\mathcal{T}$ according to the complexity of their associated decision problem.

In 1998, P. Jeavons brought to light an interesting connection linking CSP's and universal algebra [18]. Given a finite structure $\mathcal{T}$, let $\mathbb{A}(\mathcal{T})$ denote the algebra with the same universe as $\mathcal{T}$ and whose basic operations are all those that preserve the basic relations of $\mathcal{T}$. Roughly speaking, Jeavons' result states that if the two structures $\mathcal{T}$ and $\mathcal{T}'$ are such that the algebras $\mathbb{A}(\mathcal{T})$ and $\mathbb{A}(\mathcal{T}')$ are term equivalent, then the decision problems for $\mathcal{T}$ and $\mathcal{T}'$ are polynomial-time equivalent. In [5] it is shown that in fact we may restrict our attention to *idempotent* algebras: for every finite structure $\mathcal{T}$, there exists a finite structure $\mathcal{T}'$ such that (i) the decision

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problems for $\mathcal{T}$ and $\mathcal{T}'$ are polynomial-time equivalent, and (ii) the algebra $\mathbb{A}(\mathcal{T}')$ is idempotent. Bulatov, Krokhin and Jeavons have made the following conjecture (which we reformulate slightly):

Conjecture 1.1 ([5])**.** *Let $\mathcal{T}$ be a finite structure such that $\mathbb{A}(\mathcal{T})$ is an idempotent algebra. If $\mathbb{A}(\mathcal{T})$ admits a Taylor term, then there exists a polynomial-time algorithm that decides for any finite structure $\mathcal{S}$ similar to $\mathcal{T}$ whether there is a homomorphism from $\mathcal{S}$ to T.*

It is known that if the idempotent algebra $\mathbb{A}(\mathcal{T})$ admits no Taylor term then the decision problem for $\mathcal{T}$ is $\mathcal{N}P$-complete ([5], see also [23].) In particular, verification of the conjecture would solve the above-mentioned dichotomy problem. Partial results have been obtained in the other direction; noteworthy are the recent results of A. Bulatov proving tractability of the decision problem for structures admitting a Malt'sev operation [2] and dichotomy for structures on 3 elements [3].

Given a finite poset $\mathbf{P}$, one may consider the following decision problems:

(a) the *CSP for* $\mathbf{P}$: the decision problem for the structure $\mathcal{M}$ with universe the base set of the poset $\mathbf{P}$, and whose relations are (i) the ordering of $\mathbf{P}$ and (ii) every unary relation $\{p\}$ with $p \in P$;

(b) the *retraction problem for* $\mathbf{P}$: given a poset $\mathbf{Q}$ that contains a copy of $\mathbf{P}$, does $\mathbf{Q}$ retract onto $\mathbf{P}$?

These problems were shown to be polynomial-time equivalent in [20] (see also [23].) Moreover, in [14], Feder and Vardi show that for every finite structure $\mathcal{T}$ there exists a finite poset $\mathbf{P}$ such that the decision problem for $\mathcal{T}$ is polynomial-time equivalent to the retraction problem for $\mathbf{P}$. Notice that the algebra $\mathbb{A}(\mathcal{M})$ for problem (a) is idempotent; in fact its term operations are precisely the idempotent operations that preserve the ordering of $\mathbf{P}$. Hence if the above conjecture is true, then the retraction problem for a poset $\mathbf{P}$ must be decidable in polynomial time if $\mathbf{P}$ admits a Taylor term operation. This is what led us to investigate these posets, and although our findings do not seem to have any direct bearing on the conjecture yet, we believe that we have gained some important insights into their structure.

In the next section, we present some terminology and notation as well as a few basic results we shall require. Every poset may be viewed as a topological space, in such a way that order-preserving maps are precisely the continuous ones. Theorem 3.2, the main result of section 3, states that if there exists an operation on a finite connected poset which obeys Taylor identities, then all the homotopy groups of the poset are trivial.

We note in passing the following interesting corollary. In 1985, E. Corominas conjectured (see [8]) that if a poset P does not possess the fixed-point property but all its proper retracts do, then this poset is *idempotent trivial*, i.e., no idempotent operation other than projections preserves the ordering of P.[1] If a poset P has trivial homotopy groups, then the associated simplicial complex is contractible (see section 2) and hence it has the fixed-point property. It follows from a result of Baclawski and Björner [1] that P must have the fixed-point property. Hence we obtain the following special case of Corominas' (still open) conjecture: *if a poset P does not have the fixed-point property, then it admits no Taylor term.*

In section 4, we give a series of omitting types theorems for locally finite varieties analogous to those found in Chapter 9 of Hobby and McKenzie's *Structure of Finite Algebras* [17], and similar in their topological flavour to the 1977 result of W. Taylor where he introduced the terms we study here [31]. More precisely, we say that the poset P is *in the variety* $\mathcal{V}$ if there exists an algebra in $\mathcal{V}$ whose basic operations preserve the ordering of P (which presupposes of course that the base sets of $\mathbb{A}$ and P are the same.) As a partial converse to Theorem 3.2, we prove in Theorem 4.2 that an idempotent locally finite variety $\mathcal{V}$ omits type 1 if and only if every finite connected poset in $\mathcal{V}$ has trivial homotopy groups. A similar statement in terms of finite topological spaces is also presented. In Theorem 4.4, idempotent locally finite varieties omitting types 1 and 5 are characterised by the property that the finite conected posets in the variety are dismantlable. Finally, Theorem 4.5 states that idempotent locally finite varieties omitting types 1, 4 and 5 are precisely those which have no non-trivial posets.

In section 5 we conclude with a series of examples to illustrate several types of Taylor terms that can occur on posets. In particular, we construct a family of posets which admit a semilattice term, some of which are not dismantlable. In topological terms, these finite spaces are not contractible, and hence by a result of Stong are not H-spaces [30]; on the other hand, the existence of a semilattice term confirms a conjecture of W. Taylor who asked in [32] (see also [33]) whether there exist topological spaces admitting continuous operations satisfying non-trivial identities but no H-space structure.

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2. Background

We begin this section by reviewing some facts about finite posets and topological spaces we shall require later. For basic notions of topology and basic notions on

[1] A bolder conjecture in the same paper, namely that connected, ramified, directly indecomposable posets are idempotent trivial, was shown to be false by McKenzie (see [16].)

partial order and order-preserving maps we refer the reader to [6, 29] and [12, 24] respectively.

From this point on we use boldface capital letters to denote a poset, and when there is no risk of confusion we use the same symbol '$\leq$' to denote the partial order on different posets. If $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ ... are posets then P, Q, R, ... denote their respective underlying set. For our purposes a function $f\colon \mathbf{P} \to \mathbf{Q}$ is a *map* if it is order-preserving, i.e., if $f(x) \leq f(y)$ whenever $x \leq y$. If $f\colon \mathbf{P}^n \to \mathbf{P}$ is a map, then we say that the poset $\mathbf{P}$ *admits* the n-ary operation f or that f is *an operation on* $\mathbf{P}$. As usual, an operation f is *idempotent* if $f(x, \dots, x) = x$ for all x. A *subposet* $\mathbf{Q}$ of a poset $\mathbf{P}$ is a subset Q of P together with the restriction of the order relation on $\mathbf{P}$ to Q.

An element of a poset $\mathbf{P}$ is *irreducible* if it possesses either a unique upper cover or a unique lower cover in $\mathbf{P}$. Let $\mathbf{Q}$ be a subposet of $\mathbf{P}$. We say that $\mathbf{P}$ *dismantles to* $\mathbf{Q}$ if we can write $P = \{x_1, \dots, x_n\}$ such that for some j we have (i) $\mathbf{Q} = \{x_j, \dots, x_n\}$ and (ii) for all $i = 1, \dots, j-1$, x_i is an irreducible element in the subposet of $\mathbf{P}$ induced by $\{x_i, \dots, x_n\}$. In other words, one may define this concept recursively as follows: (i) $\mathbf{P}$ dismantles to $\mathbf{P}$; (ii) if $\mathbf{Q}$ is obtained from $\mathbf{P}$ by removing an irreducible element, then $\mathbf{P}$ dismantles to $\mathbf{Q}$; (iii) if $\mathbf{P}$ dismantles to $\mathbf{Q}$ and $\mathbf{Q}$ dismantles to $\mathbf{Q}'$ then $\mathbf{P}$ dismantles to $\mathbf{Q}'$. A finite poset $\mathbf{P}$ is called *dismantlable* if $\mathbf{P}$ dismantles to a one element subposet. A connected poset is *ramified* if it contains more than one element and contains no irreducibles.

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. The poset $\mathbf{P}^{\mathbf{Q}}$ consists of all order-preserving maps from $\mathbf{Q}$ to $\mathbf{P}$ ordered pointwise, i.e., $f \leq g$ if $f(q) \leq g(q)$ for all $q \in Q$. A subposet $\mathbf{R}$ of $\mathbf{P}$ is a *retract* of $\mathbf{P}$ if there exists an isotone map $r \in \mathbf{P}^{\mathbf{P}}$ such that $r(P) = R$ and $r^2 = r$. Such a map is called a *retraction* from $\mathbf{P}$ to $\mathbf{R}$. Notice that if $\mathbf{P}$ dismantles to $\mathbf{Q}$ then $\mathbf{Q}$ is a retract of $\mathbf{P}$.

Lemma 2.1 ([24]). *Let* $\mathbf{P}$ *be a finite poset. Then the following conditions are equivalent:*

(1) $\mathbf{P}$ *is ramified.*

(2) *Every automorphism of* $\mathbf{P}$ *is alone in its connected component of* $\mathbf{P}^{\mathbf{P}}$.

We shall need the following easy corollary:

Corollary 2.2 ([27]). *Let* $\mathbf{P}$ *be a finite, connected, ramified poset.*
Let $f\colon \mathbf{P}^{n+1} \longrightarrow \mathbf{P}$ *be an idempotent map on* $\mathbf{P}$ *which is not a projection. Fix elements* $a_1, \dots, a_n$ *in* $\mathbf{P}$. *Then the map* μ *defined by* $\mu(x) = f(a_1, \dots, a_n, x)$ *is not onto.*

Proof. It is easy to see that the map f induces a map $g\colon \mathbf{P}^n \to \mathbf{P}^{\mathbf{P}}$ where $g(a_1, \dots, a_n)$ is the map defined by $g(a_1, \dots, a_n)(x) = f(a_1, \dots, a_n, x)$ for all a_i and x. Since $\mathbf{P}$ is connected, so is the image of g. Suppose for a contradiction that

$g(a_1, \ldots, a_n)$ is onto for some a_i. Then it is an automorphism of $\mathbf{P}$, and by the last result it is alone in its component of $\mathbf{P}^{\mathbf{P}}$. Hence f does not depend on its first n variables, and since it is idempotent, we get that f is the projection on the last variable, a contradiction. $\square$

Let $\mathbf{P}$ be a poset. The *ideal topology* on $\mathbf{P}$ is the topology whose open sets are exactly the *order ideals* (*down-sets, initial segments*) of $\mathbf{P}$, i.e., those subsets U that satisfy the following condition: if $u \leq v$ and $v \in U$ then $u \in U$. It is easy to see that a map between posets is order-preserving if and only if it is continuous when the posets are given the ideal topology (see for example [30].) The space $\mathbf{P}$ is connected if and only if it is arc-connected, if and only if $\mathbf{P}$ is connected as a poset.

We shall use the following definition of homotopy groups (see [6]). Let I^n denote the n-fold product of the interval $[0, 1]$ with itself, and as usual let ∂I^n denote its boundary, i.e., all n-tuples from I^n that contain the entry 0 or 1. Let $\mathbf{P}$ be a poset and let $p_0 \in P$. Let S denote the collection of all continuous maps from the pair $(I^n, \partial I^n)$ to the pair $(\mathbf{P}, p_0)$, i.e., maps $f\colon I^n \longrightarrow \mathbf{P}$ such that $f(\partial I^n) \subseteq \{p_0\}$. Two maps $f, g \in S$ are *homotopic (relative to ∂I^n)* if there exists a continuous map $\phi\colon I \times I^n \longrightarrow \mathbf{P}$ such that $\phi(0, x) = f(x)$ and $\phi(1, x) = g(x)$ for all $x \in I^n$, and $\phi(t, x) = p_0$ for all t and all $x \in \partial I^n$. This is an equivalence relation on S; let $[I^n, \partial I^n; P, p_0]$ denote the collection of equivalence classes. As usual, if $f \in S$ we denote the homotopy class of f by $[f]$.

For $n \geq 1$, the *n-th homotopy group of $\mathbf{P}$ with base point p_0* is $[I^n, \partial I^n; P, p_0]$ together with the following product: if $f, g \in S$ define

$$(f \star g)(t_1, \ldots, t_n) = \begin{cases} f(t_1, \ldots, t_{n-1}, 2t_n) & \text{if } t_n \leq 1/2, \\ g(t_1, \ldots, t_{n-1}, 2t_n - 1) & \text{if } t_n \geq 1/2. \end{cases}$$

the group operation is then (well) defined by $[f] \cdot [g] = [f \star g]$. The n-th homotopy group of a connected poset $\mathbf{P}$ does not depend on the base point (up to isomorphism), and is denoted by $\pi_n(\mathbf{P})$. A poset $\mathbf{P}$ is 0-*connected* if it is connected; for $n \geq 1$ a connected poset $\mathbf{P}$ is *n-connected* if $\pi_k(\mathbf{P})$ is trivial for all $1 \leq k \leq n$. A poset is *simply connected* if it is 1-connected.

Let X and Y be path-connected topological spaces and let $f\colon X \longrightarrow Y$ be a continuous map. Then for every $n \geq 1$ f induces a group homomorphism $f^*\colon \pi_n(X) \longrightarrow \pi_n(Y)$ defined in the obvious way: $f^*([\gamma]) = [f \circ \gamma]$ for all $[\gamma] \in [I^n, \partial I^n; X, x_0]$. The map f is a *weak homotopy equivalence* if f^* is an isomorphism for all $n \geq 1$. The map f is a *homotopy equivalence* if there exists a map $g\colon Y \longrightarrow X$ such that $f \circ g$ is homotopic to the identity map on Y and $g \circ f$ is homotopic to the identity map on X. In particular, a space is *contractible* if it is homotopically equivalent to a one-point space.

There is another topological space naturally associated to a poset which we shall not require directly, but will make certain results easier to formulate (see for example [7]). Given a poset $\mathbf{P}$, consider the simplicial complex that consists of all the chains (totally ordered subsets) of $\mathbf{P}$, and let $\widehat{\mathbf{P}}$ denote the geometric realisation of this complex. There is the following striking relationship between the two spaces:

Theorem 2.3 ([25]). *Let $\mathbf{P}$ be a connected poset. There is a weak homotopy equivalence between the space $\mathbf{P}$ (with the ideal topology) and the space $\widehat{\mathbf{P}}$.*

In particular, if $\mathbf{P}$ is n-connected for all n, then $\widehat{P}$ is a contractible space. Notice however that $\mathbf{P}$ might not be contractible itself when viewed as a finite topological space; such a space is contractible if and only if the poset $\mathbf{P}$ is dismantlable [30].[2] It should also be noted that if $\mathbf{P}$ dismantles to $\mathbf{Q}$ then $\mathbf{Q}$ is homotopically equivalent to $\mathbf{P}$ [30].

Let $i \geq 1$ be an integer. In what follows, $\mathbf{i}$ will stand for the i-element antichain, i.e., the poset on i elements with no comparabilities. Let $+$ indicate the ordinal sum of posets, i.e., if $\mathbf{P}$ and $\mathbf{Q}$ are posets then $\mathbf{P}+\mathbf{Q}$ is the poset on $P \cup Q$ where $x \leq y$ if (i) $x \leq y$ in $\mathbf{P}$ or (ii) $x \leq y$ in $\mathbf{Q}$ or (iii) if $x \in P$ and $y \in Q$.

Example. Let $n \geq 1$ and consider the poset $\mathbf{S}_{n-1} = \mathbf{2} + \cdots + \mathbf{2}$ that consists of the ordinal sum of n copies of the two-element antichain. Then $\widehat{\mathbf{S}_{n-1}}$ is homeomorphic to the sphere S^{n-1}. In particular, the 4-*crown* $\mathbf{S}_1 = \mathbf{2} + \mathbf{2}$ has the following homotopy groups: $\pi_1(\mathbf{S}_1) = \mathbb{Z}$ and $\pi_k(\mathbf{S}_1) = 0$ for all $k \geq 2$. Notice also that the 4-crown is *idempotent trivial* (*projective*), i.e., the only idempotent operations on this poset are the projections [11], [8].

3. Posets with Taylor terms

Definition. Let $\mathbf{P}$ be a poset. A *Taylor term for* $\mathbf{P}$ is an order-preserving, idempotent n-ary operation f on $\mathbf{P}$ that satisfies, for each $1 \leq i \leq n$, an identity of the form

$$f(u_1, \ldots, u_{i-1}, x, u_{i+1}, \ldots, u_n) \approx f(v_1, \ldots, v_{i-1}, y, v_{i+1}, \ldots, v_n)$$

where each u_j and v_j is in $\{x, y\}$.

It is easy to see that, if a poset $\mathbf{P}$ admits a Taylor term f, then every retract of $\mathbf{P}$ admits a Taylor term (which satisfies any linear identity that f does): if r is a retraction of $\mathbf{P}$ onto $\mathbf{R}$ and f is an n-ary Taylor term on $\mathbf{P}$, then the restriction of $r \circ f$ to $\mathbf{R}^n$ is the desired operation.

If $\mathbf{P}$ is a poset, let $\mathbb{A}(\mathbf{P})$ denote an algebra with universe $\mathbf{P}$ whose fundamental operations are all the order-preserving operations on $\mathbf{P}$. The *idempotent reduct*

[2] See Figure 2 in section 5 for an example of a ramified poset $\mathbf{P}$ such that $\widehat{\mathbf{P}}$ is contractible.

of $\mathbb{A}(\mathbf{P})$ is the algebra whose fundamental operations are the *idempotent* order-preserving operations on $\mathbf{P}$. Let $\mathcal{V}(\mathbf{P})$ denote the variety generated by $\mathbb{A}(\mathbf{P})$. Finally let $\mathcal{V}_I(\mathbf{P})$ denote the variety generated by the idempotent reduct of $\mathbb{A}(\mathbf{P})$.

A *topological algebra* is a triple $\langle A; F, T\rangle$ where $\mathbb{A} = \langle A, F\rangle$ is an algebra and T is a topology on A that makes all the operations in F continuous (where the topology on powers of $\mathbb{A}$ is the usual product topology.) Let $\mathcal{V}$ be a variety. A *topological space in* $\mathcal{V}$ is a topological space $\langle A, T\rangle$ such that there exists a topological algebra $\langle A; F, T\rangle$ with $\langle A; F\rangle \in \mathcal{V}$.

We need the following special case of a result of Taylor:

Theorem 3.1 ([31]). *Let $\mathcal{V}$ be an idempotent variety. Then the following are equivalent:*

(1) *$\mathcal{V}$ admits a Taylor term;*
(2) *the arc components of every topological algebra in $\mathcal{V}$ have an Abelian fundamental group.*

We may now state and prove the main result of this section:

Theorem 3.2. *Let $\mathbf{P}$ be a connected finite poset. If $\mathbf{P}$ admits a Taylor term, then $\mathbf{P}$ is n-connected for all $n \geq 0$.*

Proof. Fix $n \geq 1$ and suppose for a contradiction that $\pi = \pi_n(\mathbf{P})$ is not trivial for some connected poset $\mathbf{P}$ admitting a Taylor term. Let $\mathbf{P}$ be a counterexample of minimal cardinality. Let $p_0 \in P$.

Claim 1. π is Abelian.

Proof of Claim 1. Since higher homotopy groups are Abelian, it suffices to consider only the case $n = 1$. The poset $\mathbf{P}$, when viewed as a finite topological space, is a topological space in $\mathcal{V}_I(\mathbf{P})$. Hence by Taylor's theorem the fundamental group $\pi_1(\mathbf{P})$ of $\mathbf{P}$ is Abelian.

In what follows, we shall denote the group π additively, i.e., its group operation will be denoted by $+$ and its neutral element by 0. Let $End(\pi)$ denote the ring of endomorphisms of π, where as usual 1 denotes the identity endomorphism and 0 denotes the constant endomorphism.

Claim 2. (i) The poset $\mathbf{P}$ is ramified; (ii) if $\mathbf{R}$ is a proper retract of $\mathbf{P}$ then $\pi_n(\mathbf{R})$ is trivial.

Proof of Claim 2. (i) If $\mathbf{P}$ is not ramified then we may dismantle it to a proper retract $\mathbf{R}$, which is homotopically equivalent to $\mathbf{P}$. In particular, $\pi_n(\mathbf{R})$ is non-trivial. However $\mathbf{R}$ admits a Taylor term because it is a retract of $\mathbf{P}$, and this contradicts the minimality of $\mathbf{P}$. The proof of part (ii) is similar.

Let $f\colon \mathbf{P}^k \to \mathbf{P}$ be a map such that $f(p_0, \dots, p_0) = p_0$. Then we define a function $f^\star\colon \pi^k \to \pi$ by the following: if $[\gamma_i] \in [I^n, \partial I^n; P, p_0]$ for $i = 1, \dots, k$, then let

$$f^\star([\gamma_1], \dots, [\gamma_k]) = [f(\gamma_1, \dots, \gamma_k)]$$

where as usual

$$f(\gamma_1,\ldots,\gamma_k)(t_1,\ldots,t_n)=f(\gamma_1(t_1,\ldots,t_n),\ldots,\gamma_k(t_1,\ldots,t_n))$$

for all $t_j \in I^n$.

Claim 3. For any map f as above, the function $f^\star$ is a well-defined group homomorphism. Furthermore, the correspondence $f \mapsto f^\star$ commutes with composition, i.e., we have

$$(f(g_1,\ldots,g_s))^\star = f^\star(g_1^\star,\ldots,g_s^\star)$$

for all maps $f, g_1,\ldots,g_s$ for which the expression is defined. Also, if f is a projection then so is $f^\star$.

Proof of Claim 3. This is a simple exercise left to the reader (see also Lemmas 6 and 8 of [33].)

Since $\mathbf{P}$ admits a Taylor term, it admits a non-trivial idempotent operation. It follows that $\mathbf{P}$ must also admit a non-trivial binary idempotent operation, call it f [22]. By Claim 1 and Claim 3 we have that, for any $u, v \in \pi$

$$\begin{aligned} f^\star(u,v) &= f^\star(u,0)+f^\star(0,v)\\ &= r_1u+r_2v \end{aligned}$$

for some $r_1, r_2 \in End(\pi)$.

Claim 4. $r_1 + r_2 = 1$ in $End(\pi)$.

Proof of Claim 4. Let $[\gamma] \in [I^n, \partial I^n; P, p_0]$. Since f is idempotent we have that

$$f^\star([\gamma],[\gamma]) = [f(\gamma,\gamma)] = [\gamma]$$

hence $f^\star$ is also idempotent. It follows that for any $u \in \pi$ we have

$$(r_1+r_2)(u) = r_1u + r_2u = f^\star(u,u) = u$$

and the claim follows.

Now consider the following maps on $\mathbf{P}$: let $f_1(x) = f(x, p_0)$ and let $f_2(x) = f(p_0, x)$ for all $x \in P$.

Claim 5. $f_1^\star = r_1$ and $f_2^\star = r_2$.

Proof of Claim 5. If ι stands for the identity map on $\mathbf{P}$, and $\overline{p_0}$ stands for the unary constant map with value p_0 on $\mathbf{P}$, then clearly $\iota^\star = 1$ and $\overline{p_0}^\star = 0$. It follows from Claim 3 that

$$\begin{aligned} f_1^\star &= f(\iota,\overline{p_0})^\star\\ &= f^\star(\iota^\star,\overline{p_0}^\star)\\ &= r_1(1)+r_2(0)\\ &= r_1. \end{aligned}$$

The proof for $i = 2$ is identical.

Claim 6. r_1 and r_2 are nilpotent elements of $End(\pi)$.

Proof of Claim 6. It suffices to prove the claim for r_1. By Lemma 2.2 the map f_1 is not onto, and thus there exists some positive integer k such that $(f_1)^k$ is a retraction onto a proper retract $\mathbf{R}$ of $\mathbf{P}$. By Claim 2 the n-th homotopy group of $\mathbf{R}$ is trivial. It follows that $(f_1)^k(\gamma)$ is homotopic to the constant map $\overline{p_0}(\gamma)$ for any $[\gamma] \in [I^n, \partial I^n; P, p_0]$. Hence by Claims 3 and 5 we have that

$$(r_1)^k = (f_1^\star)^k = ((f_1)^k)^\star = \overline{p_0}^\star = 0.$$

Now it follows from Claims 4 and 6 that r_2 is an invertible element of $End(\pi)$: indeed, if $(r_1)^k = 0$ then

$$r_2(1 + r_1 + r_1^2 + \cdots + r_1^{k-1}) = (1 - r_1)(1 + r_1 + r_1^2 + \cdots + r_1^{k-1}) = 1 - r_1^k = 1.$$

But then we have that r_2 is both nilpotent and invertible, which means that $0 = 1$ in $End(\pi)$, a contradiction since π is non-trivial. $\square$

The converse of this result is easily seen to be false. Indeed, any bounded poset (i.e., with a greatest and a least element) is dismantlable, and dismantlable posets are contractible spaces; in particular they are n-connected for all n. However, it is easy to construct bounded posets that do not admit Taylor terms. For example, consider the poset

$$\mathbf{1} + \mathbf{2} + \mathbf{2} + \mathbf{2} + \mathbf{2} + \mathbf{1};$$

every idempotent operation on this poset must preserve the 4 middle elements which form a 4-crown, and hence no idempotent operation on $\mathbf{P}$ can obey any non-trivial identity. Nonetheless, we prove in Theorem 4.2 that in an idempotent locally finite variety $\mathcal{V}$, if all the connected orderings invariant under the fundamental operations of any finite algebra in $\mathcal{V}$ are simply connected then there is a term of $\mathcal{V}$ which obeys some set of Taylor identities.

4. Omitting types theorems

We now present some omitting-types theorems based on the structural properties of the orderings that are preserved by finite algebras in the variety. The basic results and definitions we use can be found in [17]. Note also that our results can be stated for arbitrary locally finite varieties by noting that a variety omits the type set $\{1\}$ or $\{1, 5\}$ or $\{1, 4, 5\}$ if and only if its full idempotent reduct does.

An *ordered algebra* is a triple $\langle A; F, \leq \rangle$ where $\mathbb{A} = \langle A, F \rangle$ is an algebra and $\leq$ is a partial order relation on A which is invariant under all the operations in F. Let $\mathcal{V}$ be a variety. A *poset in* $\mathcal{V}$ is a poset $\mathbf{P} = \langle A, \leq \rangle$ such that there exists an ordered algebra $\langle A; F, \leq \rangle$ with $\langle A; F \rangle \in \mathcal{V}$.

To every finite topological space X is associated naturally a quasiorder relation $\trianglelefteq$ as follows: let $x \trianglelefteq y$ if every open set that contains y also contains x. This relation is clearly reflexive and transitive, and is antisymmetric precisely when X satisfies the T_0 separation axiom. It is a simple exercise to verify that a map between finite spaces is continuous if and only if it is a homomorphism for the respective quasiorders. Let $\sim$ denote the equivalence relation on X defined by $x \sim y$ if $x \trianglelefteq y \trianglelefteq x$. The relation $\trianglelefteq$ induces a partial ordering on the quotient space $X/{\sim}$; furthermore, the quotient topology is precisely the ideal topology of this partial order. We shall require the following:

Theorem 4.1 ([25], Theorem 4)**.** *Let X be a finite topological space, let $\trianglelefteq$ denote the associated quasiorder and $\sim$ the equivalence induced by $\trianglelefteq$. Then the quotient map $X\colon \longrightarrow X/{\sim}$ is a homotopy equivalence.*

Here is our first main result of this section:

Theorem 4.2. *Let $\mathcal{V}$ be a locally finite idempotent variety. Then the following statements are equivalent:*

(1) $1 \notin \text{typ}\{\mathcal{V}\}$*;*
(2) *every finite connected poset in $\mathcal{V}$ is n-connected, for all $n \geq 0$;*
(3) *for every finite connected poset $\mathbf{P}$ in $\mathcal{V}$, the space $\widehat{\mathbf{P}}$ is contractible;*
(4) *every finite path-connected topological space in $\mathcal{V}$ is n-connected, for all $n \geq 0$; is simply connected;*

Proof. (1) $\Rightarrow$ (2): Let $\mathbf{P}$ be a finite connected poset in $\mathcal{V}$. By Lemma 9.4 and Theorem 9.6 of [17], there exists a Taylor term for $\mathcal{V}$. Hence $\mathbf{P}$ admits a Taylor term and by Theorem 3.2 every homotopy group of $\mathbf{P}$ is trivial.

(2) $\Leftrightarrow$ (3): Immediate from Theorem 2.3 and the fact that a path-connected complex is contractible precisely when all its homotopy groups are trivial.

(2) $\Rightarrow$ (4): Let X be a finite path-connected space in $\mathcal{V}$. Denote by $\mathbb{A}$ the underlying algebra. Then every fundamental operation of $\mathbb{A}$ is a continuous map, so by the remarks above these operations preserve the quasiorder relation on X. In particular, they must preserve the equivalence induced by the quasiorder and hence $\sim$ is a congruence of $\mathbb{A}$. Thus $\mathbb{A}/{\sim}$ is a topological algebra in $\mathcal{V}$ whose topology is that induced by a partial order. It is easy to see that this partial order is connected since X is path-connected, so it follows that the space $X/{\sim}$ is n-connected for all $n \geq 0$; by Theorem 4.1 we conclude that X has the same property.

(4) $\Rightarrow$ (1): Suppose that $1 \in \text{typ}\{\mathcal{V}\}$. By Theorem 9.6 of [17], the variety $\mathcal{V}$ interprets in the variety of sets; thus we can find a 4-element algebra in $\mathcal{V}$ whose fundamental operations are projections. Then we can equip this algebra with the ordering of a 4-crown to make it into an ordered algebra in $\mathcal{V}$. Since the 4-crown is connected but has a non-trivial fundamental group this completes the proof of the theorem. □

Remark. Notice that in Theorem 4.2, one could replace statements (2) and (4) by the following:

(2′) *every finite connected poset in* $\mathcal{V}$ *is simply connected;*
(4′) *every finite path-connected topological space in* $\mathcal{V}$ *is simply connected.*

To prove our second result, we shall require the following special case of Lemma 4.1 of [24]:

Lemma 4.3 ([24])**.** *Let* $\mathbf{P}$ *be a finite, connected, ramified poset whose proper retracts are dismantlable. Let* p *be a ternary order-preserving operation on* $\mathbf{P}$*.*

(1) *If* p *satisfies the identity* $p(x, y, y) = x$ *then* p *is the projection onto the first variable.*
(2) *If* p *satisfies the identities* $p(x, x, y) = p(x, y, x) = x$ *then* p *is the projection onto the first variable.*

Theorem 4.4. *Let* $\mathcal{V}$ *be a locally finite idempotent variety. Then the following statements are equivalent:*

(1) $\operatorname{typ}\{\mathcal{V}\} \cap \{1, 5\} = \emptyset$*;*
(2) *every finite connected poset in* $\mathcal{V}$ *is dismantlable;*
(3) *every finite path-connected topological space in* $\mathcal{V}$ *is contractible.*

Proof. (1) $\Rightarrow$ (2): Suppose for a contradiction that there exists a finite connected poset $\mathbf{P}$ in $\mathcal{V}$ which is not dismantlable. By Theorem 9.8 of [17], we can find ternary order-preserving operations $d_0, \dots, d_n, p, e_0, \dots, e_n$ on $\mathbf{P}$ which satisfy the following identities:

$$\begin{aligned}
&x \approx d_0(x, y, z),\\
&d_i(x, y, y) \approx d_{i+1}(x, y, y) \text{ and } e_i(x, y, y) \approx e_{i+1}(x, y, y)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{for even } i < n,\\
&d_i(x, x, y) \approx d_{i+1}(x, x, y) \text{ and } e_i(x, x, y) \approx e_{i+1}(x, x, y)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{for odd } i < n,\\
&d_n(x, y, y) \approx p(x, y, y) \text{ and } p(x, x, y) \approx e_0(x, x, y),\\
&d_i(x, y, x) \approx d_{i+1}(x, y, x) \text{ and } e_j(x, y, x) \approx e_{j+1}(x, y, x)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{for odd } i < n \text{ and even } j < n,\\
&e_n(x, y, z) \approx z.
\end{aligned}$$

Since $\mathbf{P}$ is not dismantlable, it must dismantle to a ramified retract. In fact, let $\mathbf{R}$ be a ramified retract of $\mathbf{P}$ of minimum cardinality: then every proper retract of $\mathbf{R}$ is dismantlable; and clearly $\mathbf{R}$ is connected. Furthermore, it is clear that the above operations on $\mathbf{P}$ induce operations on $\mathbf{R}$ which obey the same identities. So we may apply the last lemma to $\mathbf{R}$: we have $x \approx d_1(x, y, y)$ so d_1 is the first projection. Then we have that

$$d_2(x, x, y) \approx x \approx d_2(x, y, x)$$

so d_2 is also the first projection. Continuing in this fashion we obtain that

$$p(x, y, z) \approx x.$$

However, the same argument applied to the sequence $e_n, \ldots, e_0, p$ yields $p(x, y, z) \approx z$, a contradiction.

(2) ⇒ (3): Let X be a finite path-connected space in $\mathcal{V}$. Then as in the proof of Theorem 4.2 we may construct the quotient space $X/\!\sim$ as the underlying topology of a topological (and ordered) algebra in $\mathcal{V}$. Since $X/\!\sim$ is contractible, by Theorem 4.1, X is also contractible.

(3) ⇒ (2): Immediate.

(2) ⇒ (1): Suppose that $\mathrm{typ}\{\mathcal{V}\} \cap \{1, 5\} \neq \emptyset$. Then by Theorem 9.8 of [17], the variety $\mathcal{V}$ interprets in the variety of semilattices. Hence for every finite semilattice there exists an algebra $\mathbb{A}$ in $\mathcal{V}$ with the same base set and such that the clone of term operations of $\mathbb{A}$ is contained in the clone of the semilattice. Consequently, every poset in the variety of semilattices is a poset in $\mathcal{V}$ as well. But there exists a connected non-dismantlable finite poset that admits a semilattice term (see section 5), hence there is a non-dismantlable connected poset in $\mathcal{V}$. □

Theorem 4.5. *Let $\mathcal{V}$ be a locally finite idempotent variety. Then the following statements are equivalent:*

(1) $\mathrm{typ}\{\mathcal{V}\} \cap \{1, 4, 5\} = \emptyset$;
(2) *every poset in $\mathcal{V}$ is an antichain.*
(3) *every finite path-connected topological space in $\mathcal{V}$ is trivial.*

Proof. (1) ⇒ (2): Let $\mathbf{P}$ be a poset in $\mathcal{V}$. By Theorem 9.14 and Lemma 9.13 of [17], the variety $\mathcal{V}$ is $n+1$-permutable for some n, and hence there exist operations $p_1, \ldots, p_n$ on $\mathbf{P}$ that satisfy the following identities:

$$x \approx p_1(x, y, y)$$

$$p_i(x, x, y) \approx p_{i+1}(x, y, y) \text{ for each } i,$$

$$p_n(x, x, y) \approx y.$$

It is a simple exercise to verify that these identities imply that $\mathbf{P}$ contains no comparability [10].

(2) ⇒ (3): let X be a finite path-connected topological space in $\mathcal{V}$. Then $X/\!\sim$ is a space in $\mathcal{V}$ whose topology is that of a poset in $\mathcal{V}$; since this poset is an antichain, its topology is discrete, i.e., every set is open. Hence the $\sim$-blocks are open sets in $X/\!\sim$. Since X is path-connected it follows that there can be only one block, and so the topology on X is trivial, i.e., the only open sets are X and the empty set.

(3) ⇒ (1): If $\mathrm{typ}\{\mathcal{V}\} \cap \{1, 4, 5\} \neq \emptyset$ then by Theorem 9.14 of [17] $\mathcal{V}$ is not n-permutable for any n. Now, we apply a result of Hagemann [15] (a proof is implicit

in [21]): a variety is n-permutable for some n if and only if every compatible quasi-order of any algebra in the variety is a congruence. Thus $\mathcal{V}$ contains an algebra $\mathbb{A}$ with a compatible quasi-order q that is not a congruence, i.e., q is not symmetric. Hence, there are two elements of $\mathbb{A}$ that witness the failure of the symmetricity of q. As $\mathcal{V}$ is locally finite these elements generate a finite algebra in $\mathcal{V}$ and q restricted to this algebra is a compatible quasi-order that is not a congruence. So $\mathbb{A}$ may be assumed to be finite. Moreover, by the idempotency of $\mathcal{V}$ the quasi-order q may be assumed to be connected. Then the base set of $\mathbb{A}$ equipped with the ideal topology of q is a non-trivial finite path-connected topological space in $\mathcal{V}$. $\square$

Remark. It is unlikely that further omitting-types results (such as omitting types 1, 2 and 5) can be obtained via structural properties of the posets in the variety. Let $\mathbb{F}$ be a finite field, and let ${}_{\mathbb{F}}\mathcal{V}$ be the variety of vector spaces over $\mathbb{F}$. Then $\operatorname{typ}\{{}_{\mathbb{F}}\mathcal{V}\} \subseteq \{2\}$. On the other hand, the variety $\mathcal{B}$ of Boolean algebras satisfies $\operatorname{typ}\{\mathcal{B}\} \subseteq \{3\}$. So $\mathcal{B}$ omits $\{1, 2, 5\}$ but ${}_{\mathbb{F}}\mathcal{V}$ does not. As both varieties omit $\{1, 4, 5\}$, by Theorem 4.5 the only posets in both of them are antichains.

5. Examples

In this section we present several examples of finite posets which admit various types of Taylor terms. We start with a few basic definitions. Let X be a nonempty set and let $k \geq 2$. An idempotent operation $f\colon X^k \longrightarrow X$ is a *totally symmetric operation* (TSI for short) if it satisfies $f(x_1, \ldots, x_k) = f(y_1, \ldots, y_k)$ whenever $\{x_1, \ldots, x_k\} = \{y_1, \ldots, y_k\}$. TSI operations appear naturally in the study of the complexity of constraint satisfaction problems, see [9, 23]. Clearly any TSI is a Taylor operation. There is a simple way of building k-ary TSI operations: if $\vee$ is a semilattice term on X, then the operation defined by

$$f(x_1, \ldots, x_k) = x_1 \vee x_2 \vee \cdots \vee x_k$$

is clearly a TSI operation. It is proved in [28] that a finite poset $\mathbf{P}$ admits a TSI operation of every arity if it admits one of arity $|P|$.

Let X be a nonempty set and let $k \geq 3$. A k-ary *near-unanimity operation* (nuf for short) on X is an operation f that satisfies the identities

$$f(x, \ldots, x, y) \approx f(x, \ldots, x, y, x) \approx \cdots \approx f(y, x, \ldots, x) \approx x.$$

It is known that, for every finite poset $\mathbf{P}$, the variety $\mathcal{V}(\mathbf{P})$ is congruence-modular (equivalently, congruence-distributive) precisely when $\mathbf{P}$ admits an nuf [24]. A subset $Q \subseteq P$ of a poset $\mathbf{P}$ is called an *idempotent subalgebra of* $\mathbf{P}$ if it is invariant under all the order-preserving idempotent operations on $\mathbf{P}$. A connected poset admits an nuf if and only if all its idempotent subalgebras are dismantlable (equivalently, connected) [19]. If a poset $\mathbf{P}$ admits an nuf, then it admits TSI operations of every arity [23]. The converse is false: the poset $\mathbf{P} = \mathbf{2} + \mathbf{2} + \mathbf{1}$ admits no nuf (the minimal elements form an idempotent subalgebra which is disconnected), but $\mathbf{P}$ admits a semilattice term, and hence TSI operations of every arity [23]. We do not know if a poset which admits an nuf must also admit a semilattice term.

We shall now construct a family $\mathcal{K}$ of finite posets with the following properties: (i) every poset in the family admits a semilattice term, and (ii) a finite poset admits TSI operations of every arity if and only if it is a retract of some poset in $\mathcal{K}$. Let $\mathbf{P}$ be a finite poset. A subset $Q \subseteq P$ is said to be *convex* if for all $a, b, c \in P$, $a \leq b \leq c$ and $a, c \in Q$ imply that $b \in Q$. Let $\mathbf{C_P}$ denote the family of all nonempty convex sets of $\mathbf{P}$ with the following ordering: if X and Y are convex sets in $\mathbf{P}$, declare $X \leq Y$ if for every $x \in X$ there exists $y \in Y$ such that $x \leq y$ and for every $y \in Y$ there exists $x \in X$ such that $x \leq y$. It is clear that the one-element convex sets form a subposet of $\mathbf{C_P}$ isomorphic to $\mathbf{P}$. It is shown in [28] that this subposet is a retract of $\mathbf{C_P}$ precisely when $\mathbf{P}$ admits TSI operations of every arity.

Lemma 5.1. *For any poset* $\mathbf{P}$*, the poset* $\mathbf{C_P}$ *admits a semilattice term.*

Proof. For any subset X of $\mathbf{P}$ let $C(X)$ denote the convex hull of X, i.e., the intersection of all convex subsets of $\mathbf{P}$ that contain X. It is clear that C a closure operator on $\mathbf{P}$, so the operation defined by $X \vee Y = C(X \cup Y)$ is a semilattice operation on the set of non-empty convex sets of $\mathbf{P}$ (see [26], Theorem 2.11). It remains to show that it preserves the ordering defined above. Notice first that for any $X \subseteq P$, $C(X)$ is the set of all $y \in P$ such that there exist $x, x' \in X$ with $x \leq y \leq x'$. Indeed, this set contains X, and is easily seen to be convex, so it contains $C(X)$. Conversely, $C(X)$ is convex and contains X so it must contain any element y such that $x \leq y \leq x'$ with $x, x' \in X$.

Now let $X \leq X'$ and $Y \leq Y'$ in $\mathbf{C_P}$; we must show that $X \vee Y \leq X' \vee Y'$. Let $c \in C(X \cup Y)$. By the last remark there exists $a \in X \cup Y$ such that $c \leq a$, and it is clear that there exists $b \in X' \cup Y'$ such that $a \leq b$. Hence $c \leq b$ where $b \in C(X' \cup Y')$. The dual statement is identical. □

Proposition 5.2. *There exists a* 10*-element ramified poset* $\mathbf{R}$ *with no semilattice term, but which admits a totally symmetric idempotent operation of all arities. Furthermore, there exists a poset with a semilattice term that dismantles to* $\mathbf{R}$.

Proof. Let $\mathbf{P} = \mathbf{2} + \mathbf{2}$ be the 4-crown. Let a, b denote the minimal elements and let c, d denote the maximal elements of $\mathbf{P}$. It is easy to see that in the poset $\mathbf{C_P}$, the elements that contain both c and d may be up-dismantled to $\{c, d\}$, and by symmetry the same is true for elements containing $\{a, b\}$, down-dismantling in this case (Figure 1.) In other words, the following self-maps of $\mathbf{C_P}$

$$f(X) = \begin{cases} \{c, d\} & \text{if } \{c, d\} \subseteq X, \\ X & \text{otherwise,} \end{cases}$$

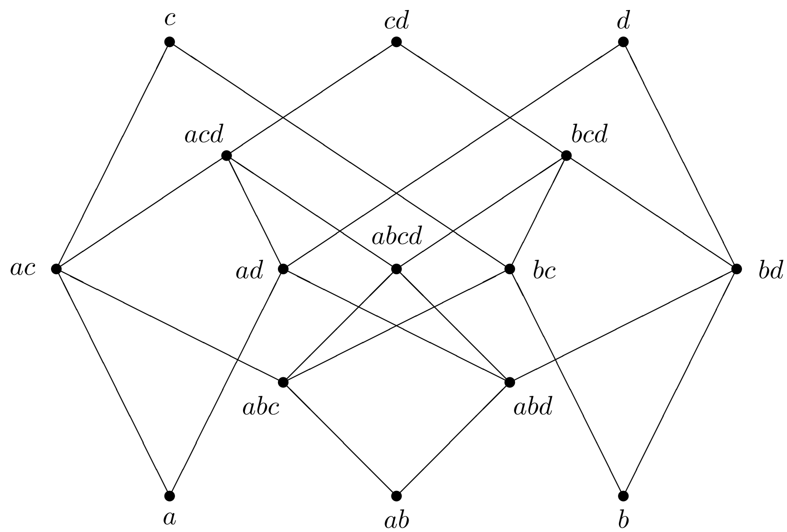

FIGURE 1. The poset $\mathbf{C_P}$ for $\mathbf{P} = \mathbf{2} + \mathbf{2}$.

and

$$g(X) = \begin{cases} \{a, b\} & \text{if } \{a, b\} \subseteq X, \\ X & \text{otherwise,} \end{cases}$$

are order-preserving retractions. One verifies immediately that $\mathbf{C_P}$ dismantles to the intersection $f(\mathbf{C_P}) \cap g(\mathbf{C_P})$ (or equivalently, $fg(\mathbf{C_P})$). Let $\mathbf{R}$ denote this poset; it consists of all subsets of $\{a, b, c, d\}$ of cardinality at most 2 (Figure 2.) We prove that $\mathbf{R}$ admits no semilattice term. Indeed, let $*$ be a commutative, idempotent order-preserving operation on $\mathbf{R}$. In what follows, we shall find it more convenient to denote the set $\{x, y\}$ by xy and the set $\{x\}$ simply by x. We have the following:

$$bc * bd = cd \text{ and } ac * bc = ab.$$

We show the first of these, the other being obtained by symmetry and duality. Consider the subposet $Q = \{ab, b, bc, bd\}$ of $\mathbf{R}$: it is isomorphic to the 4-crown. It is clear that $*$ must preserve every covering pair; furthermore, it must preserve the set $\{b, ab\}$ since these elements are precisely those below bc and bd. Since the 4-crown is idempotent trivial, $*$ cannot preserve Q, hence it cannot preserve the set $\{bc, bd\}$. Since $cd \geq bc * bd \geq b$ and $bc * bd \geq ab$, the only possibility is $bc * bd = cd$.

Now we have that

$$ac * (bc * bd) = ac * cd \in \{ac, cd\}$$

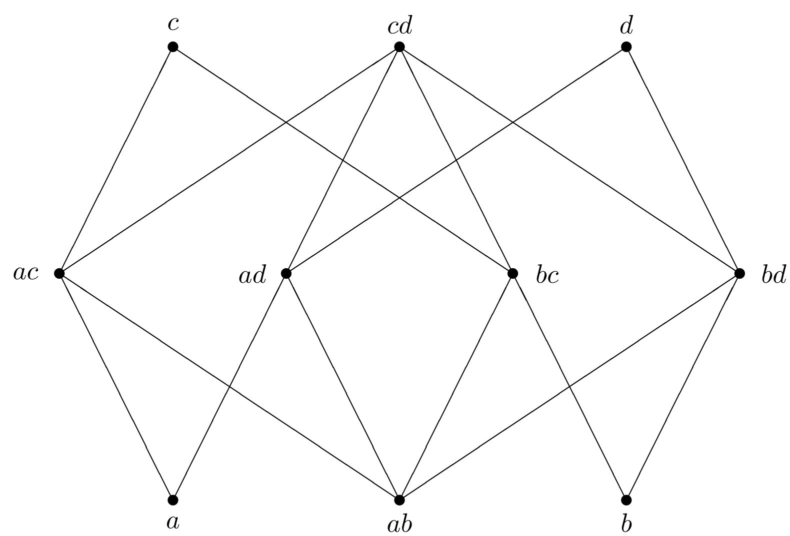

Figure 2. The poset **R**.

and

$$(ac * bc) * bd = ab * bd \in \{ab, bd\}$$

so the operation $*$ is not associative. Note that **R** is a retract of the poset $\mathbf{C_P}$ which admits a semilattice term by Lemma 5.1; in particular, both $\mathbf{C_P}$ and **R** admit totally symmetric operations of all arities. □

A topological space X is an *H-space* if there exists a continuous operation $h\colon X^2 \longrightarrow X$ and an element $e \in X$ such that both the maps $x \mapsto h(x,e)$ and $x \mapsto h(e,x)$ are homotopic to the identity on X. It is well-known that H-spaces have an Abelian fundamental group. In [32] (see also [33]) Taylor asked whether the existence of continuous operations obeying non-trivial identities force a space to be an H-space. Consider for instance the following special case: for $n \geq 2$, an n-ary operation f is an *n-mean* if f obeys the identity

$$f(x_1, \dots, x_m) \approx f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for any permutation σ of the indices. Eckmann, Ganea and Hilton have shown that if a CW-complex admits an n-mean then it must be an H-space [13]. The posets **R** and $\mathbf{C_P}$ above show that the answer to Taylor's question is negative, in general, and that the result of Eckmann, Ganea and Hilton above cannot be extended to general spaces. Indeed, it is known that the finite path-connected H-spaces are precisely the contractible ones [30]; however, **R** and $\mathbf{C_P}$ are not dismantlable (and hence not contractible spaces), but both these spaces admit Taylor terms (and in fact $\mathbf{C_P}$ admits a semilattice term). Moreover, these finite spaces admit TSI operations, and in particular, n-means.

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