ALGEBRAS, CONSTRAINT SATISFACTION AND SOLVABILITY OF SYSTEMS OF EQUATIONS

THESIS

By

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Ph.D., University of Illinois at Chicago, 1991

Submitted as partial fulfillment of the requirements
for the degree of Doctor of the Hungarian Academy of Sciences
in Mathematics in the Hungarian Academy of Sciences

2008

Contents

Introduction

The present thesis is based on publications [29], [31] and [49] of the author. The first two of them are joint papers with Benoit Larose. Section 1 of the thesis contains the basic algebraic definitions. Sections 2, 3 and 4 give an overview of the results in [29], [31] and [49], respectively. The copies of the three papers are added in the Appendix and form an essential part of the thesis.

In the past few years a new powerful tool has emerged in the investigations of constraint satisfaction problems: the theory of finite algebras. In the present introduction we briefly delineate the connection between these two areas of mathematics and discuss how our results fit into those theories. The mathematical formulation of constraint satisfaction problems was first introduced in artificial intelligence in the 1960s. Since then research in constraint satisfaction has developed rapidly, and today it has become a major area of interaction between algebra, combinatorics, logic and computer science. Algorithms based on solutions for constraint satisfaction problems are used routinely in many areas of our daily life.

We adopt the following definition due to Feder and Vardi in [16] and [17]. For a fixed finite relational structure T of finite signature the constraint satisfaction problem over T is the following decision problem denoted by $\mathcal{CSP}(T)$: *given a finite structure S similar to T, is there a homomorphism from S to T?*

The class $\mathcal{CSP}$ of problems of the form $\mathcal{CSP}(T)$ where T is a finite relational structure of finite signature is a vast subclass of $\mathcal{NP}$. It contains such problems as (Boolean) satisfiability, solvability of systems of equations, graph colorability and scheduling.

Although numerous results dealing with $\mathcal{CSP}$ appeared in the literature prior to the seminal paper [17] of Feder and Vardi from 1993, this was the first paper placing $\mathcal{CSP}$ in the algorithmic complexity context in a non-trivial way. The authors formulated several conjectures in the paper. The most important one is the dichotomy conjecture for constraint satisfaction problems:

Conjecture 1. *Each problem in $\mathcal{CSP}$ is either in $\mathcal{P}$ or is $\mathcal{NP}$-complete.*

In [17] Feder and Vardi introduced the class of bounded width problems in $\mathcal{CSP}$. These are the problems that can be solved by a certain polynomial-time local consistency algorithm or equivalently are expressible in a logical programming language called Datalog. They also introduced the problems with the ability to count, and proved that those are not of bounded width. They went on to formulate their bounded width conjecture.

Conjecture 2. *A problem in $\mathcal{CSP}$ has bounded width if and only if it simulates no problem with the ability to count.*

Later it turned out that both conjectures of Feder and Vardi correspond to certain well behaved classes of finite algebras. There is a natural connection linking $\mathcal{CSP}$ and the theory of finite algebras, first brought to light by Jeavons [22] in 1998. Given a finite structure T, let $\mathbf{A}(T)$ denote the algebra with the same underlying set as T and whose basic operations are all those that preserve the basic relations of T. Jeavons's result states that if two finite structures T and T' of finite signature are such that the algebras $\mathbf{A}(T)$ and $\mathbf{A}(T')$ are term equivalent, then the problems $\mathcal{CSP}(T)$ and $\mathcal{CSP}(T')$ are polynomial-time equivalent. In other words the clone of the term operations of $\mathbf{A}(T)$ or equivalently the variety generated by $\mathbf{A}(T)$ determines the complexity of the problem $\mathcal{CSP}(T)$ modulo polynomial time.

It is not hard to show that we may restrict our attention to structures whose related algebra is idempotent. For every finite structure T, there exists a finite structure T' such that $\mathcal{CSP}(T)$ and $\mathcal{CSP}(T')$ are polynomial-time equivalent, and the algebra $\mathbf{A}(T')$ is idempotent.

All known results which state that $\mathcal{CSP}(T)$ is tractable has the following form: the existence of some particular term operations of the algebra $\mathbf{A}(T)$ guarantees that $\mathcal{CSP}(T)$ is in $\mathcal{P}$. In other words some non-trivial set of identities over general terms determines a class of algebras (or a class of varieties), and if $\mathbf{A}(T)$ is in this class, then $\mathcal{CSP}(T)$ is a polynomial-time problem. This suggests that the investigation of such classes called Maltsev classes may play a role in characterizing the complexity of the problems in $\mathcal{CSP}$. The investigation of Maltsev classes is a subject of study in the theory of finite algebras.

Tame congruence theory grew out of universal algebra, commutator theory and lattice theory and was developed in the book [20] of Hobby and McKenzie. In Chapter 9 of [20] they gave a description of the locally finite varieties admitting a non-trivial idempotent Maltsev condition. They located five other Maltsev subclasses of those varieties, as well. These classes came in a natural way within the framework of their new theory of finite algebras. In [20] Hobby and McKenzie introduced the notion of the type set of a finite algebra and that of a variety. The type set which is a subset of the five element set $\{1, 2, 3, 4, 5\}$ whose elements are called *types* is shown to be an invariant of the algebra and the variety. The discovery of types is especially important in the study of varieties. Omittance of certain types led to Maltsev type characterizations of classes of locally finite varieties in [20]. For example omitting type 1 characterizes the locally finite varieties that admit a non-trivial idempotent Maltsev condition.

In [29] we studied the shape of compatible finite posets of locally finite varieties. Surprisingly these shapes are in close connection with omitting types. We proved the following three omitting-types theorems for a locally finite idempotent variety $\mathcal{V}$:

$\mathcal{V}$ omits type 1 if and only if the homotopy groups of every finite connected compatible poset of $\mathcal{V}$ are one element.

$\mathcal{V}$ omits types 1 and 5 if and only if every finite connected compatible poset of $\mathcal{V}$ is dismantlable.

$\mathcal{V}$ omits types 1, 4 and 5 if and only if every finite connected compatible poset of $\mathcal{V}$ is one element.

In Section 2 of the present thesis we shall give an overview of the results obtained in [29].

Suppose that T is a finite structure of finite signature. In [28] we proved that if $\mathbf{A}(T)$ is idempotent, and admits no non-trivial idempotent Maltsev condition (equivalently the variety generated by $\mathbf{A}(T)$ admits type 1), then $\mathcal{CSP}(T)$ is $\mathcal{NP}$-complete. In [8] Bulatov, Jeavons and Krokhin proved a similar result and they formulated the following strong form of the dichotomy conjecture of Feder and Vardi.

Conjecture 3. *Suppose that T is a finite structure for which the algebra $\mathbf{A}(T)$ is idempotent. Then $\mathcal{CSP}(T)$ is in $\mathcal{P}$ if the variety generated by $\mathbf{A}(T)$ omits type 1, and $\mathcal{CSP}(T)$ is $\mathcal{NP}$-complete otherwise.*

Numerous results in the literature give evidence that this conjecture holds. E.g. in the cases when T is 2-element [38], when T is an undirected graph [19], or when the algebra $\mathbf{A}(T)$ is conservative [6] the conjecture was confirmed. For further evidence see [17] for the case when $\mathbf{A}(T)$ has a near unanimity term, [7] for the case when $\mathbf{A}(T)$ admits a Maltsev term, and [3] and [21] for the case when $\mathbf{A}(T)$ is an algebra with few subpowers.

In [17] it was shown that for any problem of the form $\mathcal{CSP}(T)$ there exists a finite poset whose retraction problem is polynomial-time equivalent to $\mathcal{CSP}(T)$. This implies, see [30], that it suffices to prove the dichotomy conjecture of Feder and Vardi for $\mathcal{CSP}$ over the finite structures whose relations are the one element subsets of their base sets and a partial order. Our theorem that characterizes the locally finite idempotent varieties omitting type 1 via compatible posets may be an appropriate tool to attack this conjecture. Our other two omitting-types theorems may yield good test cases on the way to handling the conjecture.

We note that Feder and Vardi also showed that any problem of the form $\mathcal{CSP}(T)$ where T is an arbitrary finite structure is polynomial-time equivalent to a $\mathcal{CSP}$ over a digraph. Hence it suffices to prove the dichotomy conjecture for $\mathcal{CSP}$ over digraphs. By using a recent characterization [32] of Maróti and McKenzie for the class of locally finite varieties omitting type 1, Barto, Kozik and Niven managed to prove a dichotomy theorem for $\mathcal{CSP}$ over a fairly large class of digraphs, namely for the class of digraphs with no source and no sink, see [2].

In [31] we investigated the structure of bounded width problems and established a theorem involving one of the above mentioned five Maltsev classes. Let T be a finite structure of finite signature. We proved that if the algebra $\mathbf{A}(T)$ is idempotent and $\mathcal{CSP}(T)$ has bounded width, then the variety generated by $\mathbf{A}(T)$ omits types 1 and 2, and conjectured the following.

Conjecture 4. *Suppose that T is a finite structure for which the algebra $\mathbf{A}(T)$ is idempotent. Then $\mathcal{CSP}(T)$ has bounded width if and only if the variety generated by $\mathbf{A}(T)$ omits types 1 and 2.*

Our theorem yields a tool for proving that particular problems in $\mathcal{CSP}$ are not of bounded width. In [31] we show several examples of its application. We recently proved in [26] that in the idempotent case our conjecture is equivalent to the bounded width conjecture of Feder and Vardi. Partial results on the conjecture were obtained in the congruence distributive case by Kiss and Valeriote in [24] and by Carvalho, Dalmau, Marković and Maróti in [11]. In Section 3 of the thesis we shall give an overview of the results obtained in [31].

For a finite algebra $\mathbf{A}$ of finite signature the solvability problem of systems of polynomial equations over $\mathbf{A}$ is the following decision problem denoted by $\mathcal{S}ysPol(\mathbf{A})$: *given a finite system of polynomial equations S over $\mathbf{A}$, is there a solution of S over $\mathbf{A}$?*

Let $\mathcal{S}ysPol$ denote the class of problems of the form $\mathcal{S}ysPol(\mathbf{A})$ where $\mathbf{A}$ is a finite algebra of finite signature. Several dichotomy results were proved for $\mathcal{S}ysPol$ over restricted classes of algebras. For example in [18] Goldmann and Russell proved that there is a dichotomy for $\mathcal{S}ysPol$ over groups. In [25] Klíma, Tesson and Thérien verified a dichotomy for $\mathcal{S}ysPol$ over monoids and other subclasses of finite semigroups.

We already mentioned that it suffices to prove the dichotomy conjecture of Feder and Vardi for the subclass of $\mathcal{CSP}$ over digraphs or for the class of retraction problems of posets. One more item to add to this list is the class of solvability problems of systems of polynomial equations over finite algebras. In [25] Klíma, Tesson and Thérien proved that for every finite structure T, there exists an algebra $\mathbf{A}$ such that $\mathcal{CSP}(T)$ is polynomial-time equivalent to $\mathcal{S}ysPol(\mathbf{A})$.

It is not hard to prove, see [30], that for every finite algebra $\mathbf{A}$ of finite signature $\mathcal{S}ysPol(\mathbf{A})$ is polynomial-time equivalent to $\mathcal{CSP}(T)$ where the base set of T coincides with that of $\mathbf{A}$ and the relations of T are the graphs of the basic operations of $\mathbf{A}$ and the one element subsets of its base set. In [30] we exploited this correspondence as follows.

It is well known, see [45], that a finite algebra admits a non-trivial idempotent Maltsev condition if and only if it admits a specific idempotent term operation, a so called Taylor operation. Hence by a result mentioned earlier if the algebra $\mathbf{A}(T)$ is idempotent and has no Taylor term operation, then the decision problem $\mathcal{CSP}(T)$ is $\mathcal{NP}$-complete. So by the statement in the preceding paragraph we get that if an algebra $\mathbf{A}$ has no compatible Taylor operation, then $\mathcal{S}ysPol(\mathbf{A})$ is $\mathcal{NP}$-complete. This allows us to reformulate Conjecture 3 for $\mathcal{S}ysPol$ as follows.

Conjecture 5. *Let* $\mathbf{A}$ *be a finite algebra of finite signature. Then* $\mathcal{S}ysPol(\mathbf{A})$ *is in* $\mathcal{P}$ *whenever* $\mathbf{A}$ *has a compatible Taylor operation, and* $\mathcal{S}ysPol(\mathbf{A})$ *is* $\mathcal{NP}$*-complete otherwise.*

In this conjecture the dividing line between the $\mathcal{P}$ and $\mathcal{NP}$-complete cases is lost however, as far as the type set of the variety $\mathcal{V}$ generated by $\mathbf{A}$ is considered. The dividing line is shifted to the "dual" of the algebra, the algebra determined by the compatible operations of the original algebra. Interestingly enough if we put a restriction on the typeset of $\mathbf{A}$ or of the variety generated by $\mathbf{A}$, we are able to establish a dichotomy theorem for $\mathcal{S}ysPol$. This was first achieved in [30] where we proved a dichotomy theorem for $\mathcal{S}ysPol$ over the finite algebras that generate a variety omitting types 1 and 5.

The ultimate dichotomy theorem involving restrictions on the type set of the variety generated by an algebra was obtained in [49]. We proved that there is a dichotomy for $\mathcal{S}ysPol$ over the finite algebras that generate a variety omitting type 1. Both of the last two theorems encompass the cases of groups, rings, lattices, and the latter one does the case of semilattices, too. In Section 4 of the thesis we shall give an overview of the results obtained in [49].

1. Algebraic background

In this section we present the algebraic concepts that we need to state and explain the main results of the thesis. For more details we refer the reader to [10], [20] and [34].

Let A be a set. An *n-ary operation* on A is a map $f : A^n \to A$. If $n = 0$, then f is called a *constant*. In some sense the simplest operations are the projections. An n-ary operation f is called the *i-th projection* if it satisfies the identity $f(x_1, \ldots, x_i, \ldots, x_n) = x_i$. An n-ary operation f on A is *idempotent* if it satisfies the identity $f(x, \ldots, x) = x$. A subset ρ of A^m is called an *m-ary relation* on A. We say that a binary operation f *has an identity element* if there is an element e in its base set such that f satisfies the following identities $f(x, e) = f(e, x) = x$.

Let A be a finite non-empty set, let ρ be an m-ary relation on A, and let f be an n-ary operation on A; we say that f *preserves* ρ or that ρ is *closed* under f, if, given any matrix of size $m \times n$ with entries in A whose columns are elements of ρ, applying the operation f to the rows of the matrix yields a column which is in ρ. For example the projections preserve all relations on the base set.

Let f be an n-ary operation on A. The *graph* of f is defined to be the following $(n + 1)$-ary relation:

$$f^\circ = \{(a_1, \ldots, a_n, f(a_1, \ldots, a_n)) : \; a_1 \ldots, a_n \in A\}.$$

If f is a constant, then $f^\circ = \{f\}$. It is a simple exercise to verify that an operation g preserves the relation f° if and only if f preserves g°; if this is the case we say that the operations f and g *commute*. Obviously, an operation commutes with all the constants of the base set if and only if it is idempotent.

An *algebra* is a pair $\mathbf{A} = (A, \{f_i : i \in I\})$ where A is a non-empty set, I is a set and for each $i \in I$, f_i is an operation of finite arity n_i on A. The set A is called *the base set (or underlying set)* of $\mathbf{A}$ and I with the map $i \mapsto n_i$ is called *the signature* of $\mathbf{A}$. The operations f_i are called the *basic operations* of $\mathbf{A}$. The algebra $\mathbf{A}$ is *finite* if A is finite and $\mathbf{A}$ is of *finite signature* if I is finite. Two algebras are *similar* if they have the same signature. An algebra is *idempotent* if all of its basic operations are idempotent. In the thesis one encounters the following special types of algebras: unary algebras, semigroups, quasigroups, groups, monoids, semilattices, lattices, rings, modules, fields, vector spaces, Boolean algebras. For their definitions see [10] and [34].

If we are given an algebra, by using composition of functions and starting with the basic operations and the projections we can build new operations on its base set. Let $\{x_1, x_2, \ldots, x_n\}$ be a finite set of *variables.* If $\mathbf{A}$ is an algebra, an $\mathbf{A}$-*term* built from the variables $x_1, x_2, \ldots, x_n$ is defined as follows: (i) the variables $x_1, x_2, \ldots, x_n$ are $\mathbf{A}$-terms and (ii) if f is an n-ary operation symbol and $g_1, \ldots, g_n$ are $\mathbf{A}$-terms, then $f(g_1, \ldots, g_n)$ is an $\mathbf{A}$-term. Every $\mathbf{A}$-term is interpreted as a *term operation* on an algebra similar to $\mathbf{A}$ in the natural way. A *reduct* of an algebra $\mathbf{A}$ is an algebra $\mathbf{B}$ such that $\mathbf{B}$ has the same underlying set as $\mathbf{A}$ and the basic operations of $\mathbf{B}$ are term operations of $\mathbf{A}$. A *full idempotent reduct* of an algebra $\mathbf{A}$ is a reduct of $\mathbf{A}$ whose term operations are the idempotent term operations of $\mathbf{A}$.

Polynomials of $\mathbf{A}$ are defined in a similar fashion. Let C be the set of operation symbols for the constants on A. By an $\mathbf{A}$-*polynomial* built from variables $x_1, x_2, \ldots, x_n$ we mean an expression constructed as follows: (i) the variables $x_1, x_2, \ldots, x_n$ are $\mathbf{A}$-polynomials, (ii) for every $c \in C$, c is an $\mathbf{A}$-polynomial and (iii) if p is an n-ary operation symbol and q_j are $\mathbf{A}$-polynomials, then $p(q_1, \ldots, q_n)$ is an $\mathbf{A}$-polynomial. The interpretation of a polynomial in the algebra $\mathbf{A}$ is defined in a straightforward manner. We shall feel free to use the polynomial expression to denote its associated polynomial function.

Let $\mathbf{A}$ be an algebra. We say that a relation ρ on the base set of $\mathbf{A}$ is a *compatible relation* of $\mathbf{A}$ if the basic operations (equivalently, the term operations) of $\mathbf{A}$ preserve ρ. The compatible unary relations of $\mathbf{A}$ are called *subuniverses* of $\mathbf{A}$. The compatible equivalence relations of $\mathbf{A}$ are called *congruences* of $\mathbf{A}$. We say that an operation g is *compatible* with the algebra $\mathbf{A}$ if the graph of g is compatible with $\mathbf{A}$. Note that g is a compatible operation of $\mathbf{A}$ if and only if it commutes with all term operations of $\mathbf{A}$. Moreover, g is an idempotent compatible operation of $\mathbf{A}$ if and only if it commutes with all polynomial operations of $\mathbf{A}$.

A set of finitary operations on a set is called a *clone* if it contains the projections and is closed under composition. The set of all term operations or all polynomial operations of an algebra is a typical example of a clone. A clone is *idempotent* if all of its operations are idempotent.

A *relational structure* is a pair $\mathcal{T} = (T, \{r_j : j \in J\})$ where T is a non-empty set, J is a set and r_j is a relation on T of finite arity d_j, $j \in J$. The set T is called *the base set* of $\mathcal{T}$ and J with the map $j \mapsto d_j$ is called *the signature* of $\mathcal{T}$. The relations r_j are called the *basic relations* of $\mathcal{T}$. The structure $\mathcal{T}$ is *finite* if T is finite and $\mathcal{T}$ is of *finite* *signature* if J is finite. Two structures are *similar* if they have the same signature. Let $\mathcal{I} = (I, \{s_j : j \in J\})$ be a structure of signature J. A function $f : I \to T$ is a *homomorphism from* $\mathcal{I}$ *to* $\mathcal{T}$ if $f(s_j) \subseteq r_j$ for each $j \in J$. Sometimes we call such a map *relation preserving*. For any relational structure $\mathcal{T}$ the set of operations preserving the basic relations of the structure form a clone that we call *the clone of* $\mathcal{T}$. The *relational clone of* $\mathcal{T}$ is the set of all relations on T preserved by all the operations in *the clone of* $\mathcal{T}$.

Let $\mathbf{A}$ and $\mathbf{B}$ be similar algebras. $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if the base set of $\mathbf{B}$ is a subuniverse of $\mathbf{A}$ and the basic operations of $\mathbf{B}$ coincide with the restrictions of those of $\mathbf{A}$. If $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$, then $\mathbf{A}$ is called an *extension* of $\mathbf{B}$. A *homomorphism* h from $\mathbf{A}$ to $\mathbf{B}$ is a map from the base set of $\mathbf{A}$ to that of $\mathbf{B}$ such that h preserves the graphs of the basic operations. We say that $\mathbf{B}$ is a *homomorphic* image of $\mathbf{A}$ if there is a surjective homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Let $\mathbf{A}_i$, $i \in I$, be similar algebras. Their *product* is the algebra of the same signature whose base set is the Cartesian product of the base sets of $\mathbf{A}_i$, $i \in I$, and whose basic operations act componentwise as the corresponding basic operations of the $\mathbf{A}_i$.

A *variety* is a class of similar algebras which is closed under taking subalgebras, homomorphic images and products. A variety $\mathcal{V}$ is *idempotent* if the identity $f(x, \ldots, x) = x$ holds in $\mathcal{V}$ for every function symbol in the language of $\mathcal{V}$. A variety $\mathcal{V}$ is *locally finite* if every finitely generated algebra in $\mathcal{V}$ is finite. For instance, the variety $\mathcal{V}(\mathbf{A})$ *generated by a finite algebra* $\mathbf{A}$, wich consists of all homomorphic images of subalgebras of powers of $\mathbf{A}$, is locally finite. Tame congruence theory, first developed by Hobby and McKenzie in [20], is a powerful tool to study these varieties.

Let $\mathbf{A}$ be a finite algebra. If α and β are distinct congruences of $\mathbf{A}$ such that $\alpha \subset \beta$ but no congruence lies properly between them, then we say that the pair (α, β) is a *prime quotient* of congruences. In tame congruence theory, to each prime quotient is associated a *type* $i \in \{1, 2, 3, 4, 5\}$. We briefly sketch how this is done.

The starting point of the theory is to introduce a family of so-called (α, β)-*minimal* sets for each prime quotient (α, β) of congruences of $\mathbf{A}$. A unary operation r on a set A is called a *retraction*, if $r^2 = r$, in which case $r(A)$ is called a *retract* of A. We say that a subset U of $\mathbf{A}$ separates the congruences α and β if $\alpha|_U \neq \beta|_U$. It turns out that the

(α, β)-minimal sets coincide with the minimal polynomial retracts of **A** that separate α and β.

We call two algebras *polynomially equivalent* if they have the same base set and the same polynomial operations. (*Term equivalence* of algebras is defined in a similar manner.) Let U be an (α, β)-minimal set of **A**. By restricting the polynomial operations of **A** that preserve U to U we get a so called (α, β)*-minimal algebra* on U. For any fixed (α, β) the corresponding (α, β)-minimal algebras turn out to be polynomially equivalent up to isomorphism. It is a crucial fact that any (α, β)-minimal algebra induces smaller fragmental algebras, so called minimal algebras which have a very restrictive structure.

A finite algebra **A** is said to be *minimal* if every unary polynomial operation of **A** is either a constant or a permutation. A description of minimal algebras on more than two elements was given by Pálfy in [36]. By extending this description to the two element case in [20], Hobby and McKenzie proved that, up to polynomial equivalence and isomorphism, the only minimal algebras are of the following 5 types:

(1) algebras whose basic operations are permutations or constants;
(2) vector spaces;
(3) the 2-element Boolean algebra;
(4) the 2-element lattice;
(5) the 2-element semilattice.

It turns out that the minimal algebras induced by the same (α, β)-minimal algebra are polynomially equivalent up to isomorphism. Hence every prime quotient (α, β) of congruences in a finite algebra **A** has a unique *type* 1,2,3,4, or 5. The collection of all types of all prime quotients (α, β) is called the *type set* of **A**. The *typeset of a variety* is the union of all typesets of its finite members.

We say that an algebra **A** *admits a non-trivial idempotent Maltsev condition*, if there exists a finite set of identities satisfied by some idempotent term operations of **A** that is not satisfied by projections of the two element set. Most of the algebraic structures in classical algebra have this property, such as, for example, algebras with a group or semilattice term operation. The class of algebras that admit a non-trivial idempotent Maltsev condition and certain subclasses of it play a crucial role in the thesis. These classes were extensively studied in Chapter 9 of the book [20].

An n-ary operation f is a *Taylor operation* if it is idempotent and satisfies an identity of the form

$$f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) = f(y_1, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_n)$$

where $x_j, y_j \in \{x, y\}, 1 \leq j \leq n$, for every $1 \leq i \leq n$. For instance, a binary operation is a Taylor operation if and only if it is idempotent and commutative; in particular, semilattice operations are Taylor operations. Another common example of a Taylor operation is the ternary term operation $xy^{-1}z$ of a group.

Let i be an element of $\{1, 2, 3, 4, 5\}$. A finite algebra (a variety) is said to *omit type* i if its typeset does not contain type i. The connection between the typeset of a variety generated by a finite algebra and identities satisfied by the term operations of the algebra is illustrated in the following result, see Lemma 9.4 and Theorem 9.6 of [20].

Theorem 1.1. *Let* $\mathcal{V}(\mathbf{A})$ *be the variety generated by a finite algebra* $\mathbf{A}$. *Then the following are equivalent:*

(1) $\mathcal{V}(\mathbf{A})$ *omits type* 1*;*
(2) $\mathbf{A}$ *admits a nontrivial idempotent Maltsev condition;*
(3) $\mathbf{A}$ *has a Taylor term operation.*

2. Finite posets and topological spaces in locally finite varieties

We begin this section by reviewing some facts about finite posets and topological spaces we shall require later. For basic notions of topology and basic notions on partial order and order-preserving maps we refer the reader to [39] and [13], respectively.

Let P be a poset. The *ideal topology* on P is the topology whose open sets are exactly the *order ideals* (*down-sets, initial segments*) of P, i.e. those subsets U that satisfy the following condition: if $u \leq v$ and $v \in U$, then $u \in U$. We consider P as a topological space with the ideal topology. It is easy to see that a map between posets is order-preserving if and only if it is continuous when the posets are given the ideal topology (see for example [40].) The space P is connected if and only if it is arc-connected, if and only if P is connected as a poset.

We shall use the following definition of homotopy groups (see [39]). Let I^n denote the n-fold product of the interval $[0, 1]$ with itself, and as usual let ∂I^n denote its boundary, i.e. all n-tuples from I^n that contain the entry 0 or 1. Let P be a topological space and let $p_0 \in P$. Let S denote the collection of all continuous maps from the pair $(I^n, \partial I^n)$ to the pair (P, p_0), i.e. maps $f : I^n \longrightarrow P$ such that $f(\partial I^n) \subseteq \{p_0\}$. Two maps $f, g \in S$ are *homotopic (relative to ∂I^n)* if there exists a continuous map $\phi : I \times I^n \longrightarrow P$ such that $\phi(0, x) = f(x)$ and $\phi(1, x) = g(x)$ for all $x \in I^n$, and $\phi(t, x) = p_0$ for all t and all $x \in \partial I^n$. This is an equivalence relation on S; let $[I^n, \partial I^n; P, p_0]$ denote the collection of equivalence classes. As usual, if $f \in S$ we denote the homotopy class of f by $[f]$.

For $n \geq 1$, the *n-th homotopy group of P with base point p_0* is the set $[I^n, \partial I^n; P, p_0]$ together with the following product: if $f, g \in S$ define

$$(f \star g)(t_1, \ldots, t_n) = \begin{cases} f(t_1, \ldots, t_{n-1}, 2t_n) & \text{if } t_n \leq 1/2, \\ g(t_1, \ldots, t_{n-1}, 2t_n - 1) & \text{if } t_n \geq 1/2. \end{cases}$$

the group operation is then (well) defined by $[f] \cdot [g] = [f \star g]$. The n-th homotopy group of an arc-connected topological space P does not depend on the base point (up to isomorphism), and is denoted by $\pi_n(P)$. The first homotopy group $\pi_1(P)$ is called the *fundamental group* of P. It is well known that apart from the fundamental group all homotopy groups are Abelian.

Example. Let P be the *4-crown*, i.e. P is the poset $\mathbf{2} + \mathbf{2}$ where $\mathbf{2}$ stands for the two element antichain and $+$ denotes the usual ordinal sum. Then P has the following homotopy groups: $\pi_1(P_1) = \mathbb{Z}$ and $\pi_n(P_1) = 0$ for all $n \geq 2$. Notice that the only idempotent operations of the 4-crown are the projections.

A term f in the language of a variety $\mathcal{V}$ is called *idempotent* if the identity $f(x, \dots, x) = x$ holds in $\mathcal{V}$. Recall that at the end of Section 1 we introduced the definition of a Taylor operation. In a similar fashion, we say that a *variety* $\mathcal{V}$ *admits a Taylor term* if $\mathcal{V}$ has an n-ary idempotent term f such that for every $1 \leq i \leq n$ there exists an identity of the form

$$f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) = f(y_1, \dots, y_{i-1}, y, y_{i+1}, \dots, y_n)$$

where $x_j, y_j \in \{x, y\}, 1 \leq j \leq n$, that holds in $\mathcal{V}$. Such an f is called a *Taylor term* for $\mathcal{V}$.

Let $\mathcal{V}$ be a variety. A group $\mathbf{G}$ is called a *compatible group of* $\mathcal{V}$, if there is an algebra $\mathbf{A}$ in $\mathcal{V}$ such that the underlying sets of $\mathbf{G}$ and $\mathbf{A}$ are the same and the operations of $\mathbf{G}$ commute with those of $\mathbf{A}$. A *compatible poset of* $\mathcal{V}$ is a poset P such that there exists an algebra in $\mathcal{V}$ whose base set equals that of P and whose basic operations are order preserving with respect to P. (In [29], following Taylor, we used the terminology that P is poset in $\mathcal{V}$ rather than P is a compatible poset of $\mathcal{V}$.)

To prove the fundamental result of [29] we needed to use the following theorem of Taylor from [45].

Theorem 2.1. *Let $\mathcal{V}$ be a variety that admits a Taylor term. Then every compatible group of $\mathcal{V}$ is Abelian.*

It is not hard to show that the homotopy groups of a finite compatible poset of a variety $\mathcal{V}$ are compatible groups of $\mathcal{V}$. So by the preceding theorem if $\mathcal{V}$ admits a Taylor term, then the homotopy groups of a finite compatible poset of $\mathcal{V}$ are Abelian. More is true, as shown by the fundamental result of [29].

Theorem 2.2 ([29])**.** *Let $\mathcal{V}$ be a variety that admits a Taylor term. Then the homotopy groups of every finite connected compatible poset of $\mathcal{V}$ are one element.*

The proof of this theorem is based on the observation that there is a homomorphism from the clone of the order preserving operations of P that fix a designated element of P to the clone of term operations of the module $\mathbf{M}$, where $\mathbf{M}$ coincides with $\pi_n(P)$ considered as a module over its endomorphism ring.

A finite poset P *has the fixed point property* if every unary order preserving operation on P has a fixed point. By a result of Baclawski and Björner in [1] we get the following interesting corollary of the preceding theorem.

Corollary 2.3 ([29]). *If a finite connected poset admits a Taylor operation, then it has the fixed point property.*

We now present three omitting-types theorems for locally finite idempotent varieties. By the use of Theorem 2.2 we obtain our first omitting-types theorem.

Theorem 2.4 ([29]). *Let $\mathcal{V}$ be a locally finite idempotent variety. Then the following statements are equivalent:*

(1) $1 \notin \text{typ}\{\mathcal{V}\}$.
(2) *The homotopy groups of every finite connected compatible poset of $\mathcal{V}$ are one element.*

An element of a poset P is *irreducible* if it possesses either a unique upper cover or a unique lower cover in P. Let Q be a subposet of P. We say that P *dismantles to* Q if we can write $P = \{x_1, \ldots, x_n\}$ such that for some j we have $Q = \{x_j, \ldots, x_n\}$ and for all $i = 1, \ldots, j-1$, x_i is an irreducible element in the subposet of P induced by $\{x_i, \ldots, x_n\}$. A finite poset P is called *dismantlable* if P dismantles to a one element subposet. It is a well known fact, see [40], that the homotopy groups of finite dismantlable posets are one element. Dismantlable posets play an important role in our second omitting-types theorem.

Theorem 2.5 ([29]). *Let $\mathcal{V}$ be a locally finite idempotent variety. Then the following statements are equivalent:*

(1) $\text{typ}\{\mathcal{V}\} \cap \{1, 5\} = \emptyset$.
(2) *Every finite connected compatible poset of $\mathcal{V}$ is dismantlable.*

Our third omitting-types theorem is as follows.

Theorem 2.6 ([29]). *Let $\mathcal{V}$ be a locally finite idempotent variety. Then the following statements are equivalent:*

(1) $\text{typ}\{\mathcal{V}\} \cap \{1, 4, 5\} = \emptyset$.
(2) *Every finite connected compatible poset of $\mathcal{V}$ is one element.*

It is mentioned in Chapter 9 of [20] that there are six natural omitting-types conditions for locally finite varieties. By using the notion of compatible posets we gave characterizations of three out of the six conditions in the preceding theorems. By using the notions of compatible groups and compatible posets we are able to complete the picture. The following characterization handles the case of omitting types 1 and 2. Since this result did not appear in print, we supply a short proof to it.

Theorem 2.7. *Let $\mathcal{V}$ be a locally finite idempotent variety. Then the following statements are equivalent:*

(1) $\mathrm{typ}\{\mathcal{V}\} \cap \{1, 2\} = \emptyset$.
(2) *Every finite compatible group of $\mathcal{V}$ is one element.*

Proof. In the proof we use the result from [31], see Theorem 3.6 in the present thesis, which states that a locally finite idempotent variety omits types 1 and 2 if and only if it does not interpret in any variety generated by an affine algebra with at least two elements. The definitions of affine algebras and interpretation of varieties can be found in the next section prior to Theorem 3.6.

Suppose first that $\mathcal{V}$ admits type 1 or type 2. Then $\mathcal{V}$ interprets in a variety generated by an affine algebra $\mathbf{A}$ with at least two elements. Actually, by the proof of the above theorem in [31], we may assume that $\mathbf{A}$ is finite. Hence there is a finite algebra $\mathbf{B}$ in $\mathcal{V}$ such that $\mathbf{B}$ is a reduct of $\mathbf{A}$. Since $\mathbf{A}$ is affine, $x - y + z$ is a compatible operation of $\mathbf{A}$ for some Abelian group operation $x + y$ on the base set of $\mathbf{A}$. Hence $x - y + z$ is a compatible operation of $\mathbf{B}$, as well. Since $\mathbf{B}$ is idempotent the constant 0 also is a compatible operation of $\mathbf{B}$. Thus, $x + y$ is a compatible group operation of $\mathbf{B}$ and hence $\mathcal{V}$ has a finite compatible group with at least two elements.

Suppose now that $\mathbf{G}$ is a finite compatible group of $\mathcal{V}$ with at least two elements. Then there is an algebra $\mathbf{A}$ in $\mathcal{V}$ with the same underlying set as $\mathbf{G}$ such that the group operation of $\mathbf{G}$ is a compatible operation of $\mathbf{A}$. Then by Theorem 1.1 either $\mathcal{V}$ admits type 1 or $\mathcal{V}$ admits a Taylor term. In the latter case, by Theorem 2.1, the group operation of $\mathbf{G}$ is commutative and hence $\mathbf{A}$ is a reduct of an affine algebra. It is well known, see [20], that any such algebra $\mathbf{A}$ has a type set contained in $\{1, 2\}$. So if $\mathcal{V}$ has a compatible group with at least two elements, then $\mathcal{V}$ admits either type 1 or 2. □

Combining the preceding theorem and the two omitting-types theorems prior to it, similar theorems can be given for the case of omitting types 1, 2 and 5 and for the case of omitting types 1, 2, 4 and 5. So for all of the six Maltsev classes described by Hobby and McKenzie in [20] there is a characterization in terms of compatible posets and compatible groups.

Theorem 2.8. *Let $\mathcal{V}$ be a locally finite idempotent variety. Then the following statements are equivalent:*

(1) $\mathrm{typ}\{\mathcal{V}\} \cap \{1, 2, 5\} = \emptyset$.
(2) *Every finite compatible group of $\mathcal{V}$ is one element, and every finite compatible poset of $\mathcal{V}$ is dismantlable.*

Theorem 2.9. *Let $\mathcal{V}$ be a locally finite idempotent variety. Then the following statements are equivalent:*

(1) $\mathrm{typ}\{\mathcal{V}\} \cap \{1, 2, 4, 5\} = \emptyset$.
(2) *The finite compatible groups and compatible connected posets of $\mathcal{V}$ are one element.*

The *full idempotent reduct* of a variety $\mathcal{V}$ is the variety $\mathcal{W}$ whose language has a function symbol for each idempotent term of $\mathcal{V}$ and no others, and $\mathcal{W}$ consists of all algebras that satisfy all identities that hold in $\mathcal{V}$ for the idempotent terms. For example, if $\mathcal{V}$ is generated by an algebra $\mathbf{A}$, then its full idempotent reduct $\mathcal{W}$ is generated by a full idempotent reduct of $\mathbf{A}$. Note that our six omitting-types theorem can be formulated for arbitrary locally finite varieties by observing that a locally finite variety omits any of the six type sets occuring in the above theorems if and only if its full idempotent reduct does.

In the proof of Theorem 2.5, when type 1 or 5 occurs in $\mathrm{typ}\{\mathcal{V}\}$, we needed to come up with a poset that admits a semilattice operation but is not dismantlable. Such a poset is displayed in Figure 1. As an instance of a general construction in [29], the poset S in Figure 1 admits a semilattice operation. It is easy to see that S dismantles down onto the poset R in Figure 2. As R has no irreducible element, S is not dismantlable. This example allowed us to answer an open question of Taylor in [46].

A topological space X is an *H-space* if there exists a continuous operation $h : X^2 \longrightarrow X$ and an element $e \in X$ such that both the maps $x \mapsto h(x, e)$ and $x \mapsto h(e, x)$ are homotopic to the identity on X. It is well-known that H-spaces have an Abelian fundamental group. In [46] (see also [47]) Taylor asked whether the existence of continuous operations obeying non-trivial identities forces a space to be an H-space.

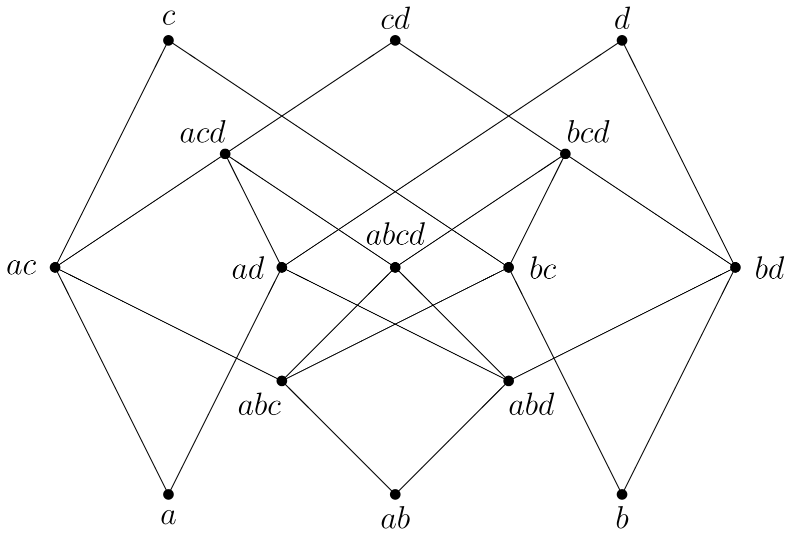

FIGURE 1. The poset S.

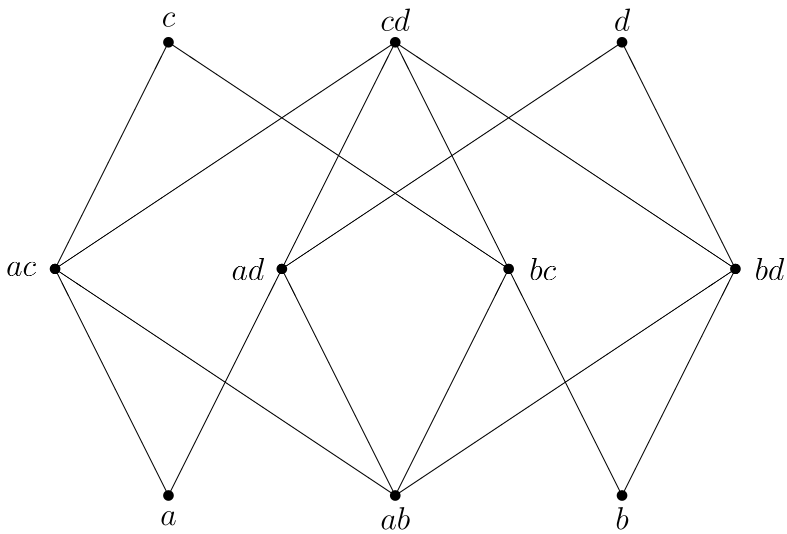

FIGURE 2. The poset R.

Consider for instance the following special case: for $n \geq 2$, an n-ary operation f is an n-*mean* if f obeys the identity

$$f(x_1, \ldots, x_m) \approx f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for any permutation σ of the indices. Eckmann, Ganea and Hilton have shown that if a CW-complex admits an n-mean, then it must be an H-space [14].

The poset S shows that the answer to Taylor's question is negative, in general, and that the result of Eckmann, Ganea and Hilton above cannot be extended to general spaces.

Indeed, as S admits a semilattice operation, it also admits an n-mean for every n, in particular admits a Taylor operation. In [40] it is shown that a finite connected poset is an H-space if and only if it is dismantlable. We already saw that S is not dismantlable, hence S is not an H-space.

3. Bounded width problems and algebras

Following Feder and Vardi [17] we give a definition of bounded width problems via a two player game. It turns out that these problems in $\mathcal{CSP}$ are solvable by a particular local consistency algorithm (in polynomial time).

Let I be a relational structure. As usual, if K is a non-empty subset of I, the *substructure induced by* K is the structure with base set K and whose relations are those of I restricted to K. Let k be a positive integer. We call the subsets (substructures) of size at most k of I the *k-subsets* (*k-substructures*) of I.

A two-player game: Let A be a finite relational structure of finite signature, and $1 \leq l < k$ integers. Let I be a relational structure similar to A. We present a two-player combinatorial game as in [17], the *(l,k)-game on I*. The game is played by the *Spoiler* and the *Duplicator* in alternating turns. In each round of the game the Spoiler selects a k-substructure K' such that $|K \cap K'| \leq l$ where K is the k-subtructure selected by the Spoiler in the preceding round. In the same round the Duplicator picks a homomorphism $f' : K' \to A$ such that $f|_{K \cap K'} = f'|_{K \cap K'}$ where $f : K \to A$ is the homomorphism picked by the Duplicator in the preceeding round. (In the first round the Spoiler is allowed to select any k-substructure of I and the Duplicator is allowed to pick any homomorphism from the k-substructure selected by the Spoiler to A.) The Spoiler wins the game if at some point the Duplicator is not able to pick a homomorphism in the way described above. As usual, we say that the Spoiler has a *winning strategy on I* if the Spoiler can play so that the Duplicator, whatever sequence of moves it makes, is eventually unable to pick a homomorphism according to the rules of the game.

A problem $\mathcal{CSP}(A)$ has *width* (l,k) if for any relational structure I for which the Spoiler has no winning strategy in the (l,k)-game, there exists a homomorphism from I to A. We say that $\mathcal{CSP}(A)$ has *width* l if it has width (l,k) for some k, and that $\mathcal{CSP}(A)$ has *bounded width* if it has width l for some l.

Given similar structures I and A and a subset K of I, we let $Hom(K,A)$ denote the set of homomorphisms from K to A where K is viewed as a substructure of I.

We now present a local consistency algorithm that leads to an equivalent definition of the bounded width problems. This definition is the one we used to prove the main results in [31]. Fix a structure A and integers $1 \leq l < k$.

(l,k)-algorithm

Input: A structure I similar to A.

Initial step: To every k-subset K of I assign a relation ρ_K that consists of all maps in $Hom(K, A)$ viewed as $|K|$-tuples;

Iteration step: Choose, if they exist, two k-subsets H and K of I with $|H \cap K| \leq l$ such that there is a map in ρ_H whose restriction to $H \cap K$ is not equal to the restriction to $H \cap K$ of any map in ρ_K, and throw out from ρ_H all such maps. If no such H and K are found, then stop and output the current relations assigned to the k-subsets of I.

We refer to the relations ρ_K obtained at the end of the algorithm as the *output relations*. Since the number of k-subsets of I is $\mathcal{O}(n^k)$ where n is the size of the instance, and in each iteration step the sum of the sizes of the relations ρ_K is decreasing, the algorithm stops in polynomial time in the size of the structure I.

Notice that the choice of the pair H and K in each iteration step of the algorithm is arbitrary. So the (l,k)-algorithm has several different versions depending on the method of the choice of the pair H and K. However, in [31] we proved the following.

Proposition 3.1 ([31])**.** *Let A and I be similar relational structures. Then any two versions of the (l,k)-algorithm for I output the same relations.*

The notions of (l,k)-game and (l,k)-algorithm are connected by the following proposition:

Proposition 3.2 ([31])**.** *Let A and I be similar relational structures. Then the (l,k)-algorithm for I yields empty output relations if and only if the Spoiler has a winning strategy in the (l,k)-game for I.*

Clearly, if the output relations of the (l,k)-algorithm for I are empty, then there is no homomorphism from I to A; however, it might be that the converse does not hold. By the last result, it follows that $\mathcal{CSP}(A)$ has bounded width if for some choice of parameters l and k the (l,k)-algorithm correctly decides the problem $\mathcal{CSP}(A)$: in particular, we get that $\mathcal{CSP}(A)$ is a polynomial-time problem. For example, any relational structure of finite type whose relations are invariant under a semilattice operation, or a near-unanimity operation, has bounded width [17]. As an application of our main result we shall give several examples of problems that do not have bounded width at the end of the section.

Let l, k, l', k' be integers such that $1 \leq l < k$ and $1 \leq l' < k'$ with $l' \geq l$ and $k' \geq k$. It can be easily verified that if $\mathcal{CSP}(A)$ has width (l, k), then it has width (l', k'). For convenience, we introduce the following terminology: a relational structure A is called an (l, k)-*structure* if $\mathcal{CSP}(A)$ has width (l, k).

In [31] we investigated the properties of the algebras associated to structures of bounded width $\mathcal{CSP}$, which prompted the following natural definition. We say that a finite algebra $\mathbf{A}$ has *bounded width* if for every relational structure A of finite signature whose base set coincides with the that of $\mathbf{A}$ and whose relations are subalgebras of finite powers of $\mathbf{A}$, the problem $\mathcal{CSP}(A)$ has bounded width.

Let T be a relational strucure and $\mathbf{A}$ an algebra. We say that $\mathbf{A}$ is an algebra for T, or equivalently T is a structure for $\mathbf{A}$, if the set of term operations of $\mathbf{A}$ coincides with the set of operations preserving the relations of T. We present a result that ensures that if $\mathbf{A}$ is an algebra for an (l, k)-structure, then it is an algebra of bounded width. Its proof, although much more involved, is similar in flavor to the proof of the following result of Jeavons [22]: *if B is a relational structure of finite type whose base set coincides with the base set of A and whose relations are in the relational clone of A, then $\mathcal{CSP}(B)$ is polynomial-time reducible to $\mathcal{CSP}(A)$.*

Lemma 3.3 ([31])**.** *Let A be an (l, k)-structure. If B is a relational structure whose base set coincides with the base set of A and whose relations are in the relational clone of A, then B is an (l', k')-structure for some l' and k'.*

A finite algebra $\mathbf{A}$ is called *locally tractable* if the problem $\mathcal{CSP}(A)$ is in $\mathcal{P}$ for every relational structure A of finite signature whose base set coincides with that of $\mathbf{A}$ and whose relations are subalgebras of finite powers of $\mathbf{A}$. It follows from results in [8] and [9] that *if a finite algebra $\mathbf{A}$ is locally tractable, then so is every finite algebra in $\mathcal{V}(\mathbf{A})$.* An analogous statement is valid for bounded width algebras:

Lemma 3.4 ([31])**.** *Every finite algebra in the variety generated by a bounded width algebra has bounded width.*

Our following result states that bounded width is preserved under interpretation of varieties; this will be used to produce a criterion which enables us to prove that certain problems in $\mathcal{CSP}$ do not have bounded width. For the purposes of Theorem 3.5 we shall define interpretability as follows: if $\mathbf{B}$ is an algebra, we say that a variety $\mathcal{V}$ *interprets in* $\mathcal{V}(\mathbf{B})$ if $\mathbf{B}$ has a reduct in $\mathcal{V}$. We note that this definition is equivalent to the following. The variety $\mathcal{V}(\mathbf{A})$ interprets in the variety $\mathcal{V}(\mathbf{B})$ if and only if there exists a clone homomorphism from the clone of term operations of $\mathbf{A}$ to the clone of term operations of $\mathbf{B}$, where a map between clones is called a *clone homomorphism* if it preserves arity, maps projections to projections and commutes with composition (see [20] page 131 for details).

Theorem 3.5 ([31])**.** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are finite algebras such that* $\mathcal{V}(\mathbf{A})$ *interprets in* $\mathcal{V}(\mathbf{B})$ *and* $\mathbf{A}$ *has bounded width, then* $\mathbf{B}$ *also has bounded width.*

As the main result of this section we present a criterion to determine if certain algebras are not of bounded width. This criterion is based on the notion of the type set of an algebra and of a variety.

In the introduction we already mentioned the result, see [8], [28]: *let* $\mathbf{A}$ *be a finite, idempotent algebra such that* $\mathcal{V}(\mathbf{A})$ *admits type 1. If* A *is a structure for* $\mathbf{A}$*, then the problem* $\mathcal{CSP}(A)$ *is* $\mathcal{NP}$*-complete.*

In [31] Theorem 3.5 was used to prove a parallel result, namely that the variety generated by an idempotent algebra of bounded width must omit types 1 and 2. On the way towards this result we needed the next lemma whose proof can be put together from results contained in [20] and [44].

A congruence θ of an algebra $\mathbf{A}$ is *Abelian* if for any n-ary polynomial f of $\mathbf{A}$, and any $u, v, x_1, \ldots, x_n, y_1, \ldots, y_n \in A$, if $u\theta v$ and $x_i\theta y_i$ for all $i = 1, \ldots, n$, then $f(u, x_1, \ldots, x_n) = f(u, y_1, \ldots, y_n)$ implies $f(v, x_1, \ldots, x_n) = f(v, y_1, \ldots, y_n)$. An algebra is *Abelian* if all its congruences are Abelian. An algebra is *affine* if its clone of polynomial operations coincides with the clone of polynomial operations of a module on the same underlying set. Affine algebras are prototypical examples of Abelian algebras.

Lemma 3.6 ([31])**.** *For a locally finite idempotent variety* $\mathcal{V}$ *the following are equivalent:*

(1) $\mathcal{V}$ *omits types* 1 *and* 2.

(2) *The only Abelian congruence of any algebra in $\mathcal{V}$ is the identity relation.*
(3) *$\mathcal{V}$ does not interpret in any variety generated by an affine algebra with at least two elements.*

In [31] we used Theorem 3.5 and Lemma 3.6 to prove our main result.

Theorem 3.7 ([31])**.** *If $\mathbf{A}$ is a finite idempotent algebra of bounded width, then $\mathcal{V}(\mathbf{A})$ omits types 1 and 2.*

The preceding theorem can be used to identify algebras that do not have bounded width even if they are not idempotent, as the next lemma shows. Given an algebra $\mathbf{A}$ and a subset B of its underlying set, let $\mathbf{A}|_{\mathbf{B}}$ denote the algebra with underlying set B whose basic operations are the restriction to B of every term operation of $\mathbf{A}$ that preserves B.

Lemma 3.8 ([31])**.**

(1) *Let $\mathbf{A}$ be a finite algebra, and let r be a unary term of $\mathbf{A}$ such that $r^2 = r$. Let $B = r(A)$. Then the algebra $\mathbf{A}|_{\mathbf{B}}$ has bounded width if and only if $\mathbf{A}$ has bounded width.*
(2) *Let $\mathbf{A}$ be an algebra whose term operations are surjective and let $\mathbf{B}$ be its full idempotent reduct. Then $\mathbf{A}$ has bounded width if and only if $\mathbf{B}$ has bounded width.*

Now we mention three applications of our bounded width criterion. Consider the following situation. Given an $\mathcal{NP}$-complete problem in $\mathcal{CSP}$, we would like to show that it is not of bounded width. The result of course follows if we assume that $\mathcal{P} \neq \mathcal{NP}$, since the bounded width problems are in $\mathcal{P}$. It may be interesting to get a proof without using the hypothesis that $\mathcal{P} \neq \mathcal{NP}$. This is what we did in our first two applications.

Let H be a fixed irreflexive and symmetric digraph. In [19], Hell and Nešetřil proved that $\mathcal{CSP}(H)$ is in $\mathcal{P}$ if H is bipartite, and it is $\mathcal{NP}$-complete otherwise. As the first application of Theorem 3.7 and Lemma 3.8, by analyzing the proof of Hell and Nešetřil we showed in [31] that the problem $\mathcal{CSP}(H)$ is not of bounded width when H is non-bipartite. In [35], Nešetřil and Zhu gave a different proof of the same result.

In [35] they also asked whether there exists a direct proof, without assuming $\mathcal{P} \neq \mathcal{NP}$, that $\mathcal{CSP}(H)$ does not have bounded width, when H is a *oriented cycle* and $\mathcal{CSP}(H)$ is $\mathcal{NP}$-complete. We used our criterion to answer Nešetřil and Zhu's question. Our result relies on a proof of Feder that for a oriented cycle H the problem $\mathcal{CSP}(H)$ is either in $\mathcal{P}$ or $\mathcal{NP}$-complete [15].

Feder shows that every oriented graph H which is a path or an unbalanced cycle admits a majority operation, and hence in this case $\mathcal{CSP}(H)$ has bounded width. A relational structure is a *core* if all of its endomorphims are automorphisms. Note that, if $\mathbf{A}$ is an algebra for a core then the term operations of $\mathbf{A}$ are surjective. Obviously if an oriented cycle H is not a core, then it retracts onto a path, and then $\mathcal{CSP}(H)$ has bounded width by Lemma 3.8. It thus suffices to consider the case where H is a core. In [31] we proved the following.

Proposition 3.9 ([31])**.** *Let H be an irreflexive, oriented cycle which is a core, and let $\mathbf{A}$ be an algebra for H.*

(1) *If $\mathcal{V}(\mathbf{A})$ admits type 1, then $\mathcal{CSP}(H)$ is $\mathcal{NP}$-complete, and it does not have bounded width.*
(2) *If $\mathcal{V}(\mathbf{A})$ omits type 1, then $\mathcal{CSP}(H)$ is in $\mathcal{P}$, and in fact has bounded width.*

In our third application we gave an example of a finite order-primal algebra that generates a variety omitting types 1 and 2. Such an example had not been known before. Strangely enough we had to use the hypothesis that $\mathcal{P} \neq \mathcal{NP}$ to prove the existence of such an algebra. We give a brief sketch of our construction below.

We define the retraction problem for a finite poset P, denoted by $\mathcal{R}et(P)$: *given a finite subposet Q of P, is there a retraction from P onto Q?* The proof of our third application was based on an analysis of a construction in [17], where Feder and Vardi associate to every relational structure B a poset P of depth 3, to prove that every problem in $\mathcal{CSP}$ is polynomial-time equivalent to a poset retraction problem. The resulting poset is called the *Feder-Vardi poset P* for the relational structure B. In [31] we proved the following theorem.

Theorem 3.10 ([31])**.** *Let B be a relational structure with a single relation and $\mathbf{B}$ an algebra for B. Let P' be the relational structure obtained from the Feder-Vardi poset P related to B by adding all one-element subsets of P as unary relations. Let $\mathbf{A}$ be an algebra for P'. Then $\mathcal{V}(\mathbf{A})$ interprets in $\mathcal{V}(\mathbf{B})$.*

The following corollary is a straightforward consequence of Theorems 3.5 and 3.10.

Corollary 3.11 ([31])**.** *Let B be a relational structure with a single relation and let P be its Feder-Vardi poset. If $\mathcal{R}et(P)$ has bounded width, then $\mathcal{CSP}(B)$ also has bounded width.*

In [31] the preceding corollary was used to get an example of a poset P such that the variety generated by an algebra for P admits type 2 but omits type 1.

Proposition 3.12 ([31])**.** *Let P be the Feder-Vardi poset of the two element structure $B = (\{0,1\}, \{(x,y,z,0) : \ x+y+z = 1\})$. The variety generated by an algebra $\mathbf{A}$ for P admits type 2 and omits type 1, provided $\mathcal{P} \neq \mathcal{NP}$.*

4. Solvability of systems of polynomial equations over finite algebras

In this section we investigate the complexity of determining if a given system of polynomial equations over a finite algebra admits a solution. Let $\mathbf{A}$ be a finite algebra of finite signature. Recall that the solvability problem of systems of polynomial equations over $\mathbf{A}$ is the problem denoted by $\mathcal{S}ysPol(\mathbf{A})$: *given a system S of polynomial equations over $\mathbf{A}$, is there a solution of S over $\mathbf{A}$?*

This problem has been studied and a dichotomy theorem has been obtained in the special cases of groups [18], monoids and some other subclasses of semigroups [25]. In [30] we adopted a new viewpoint of investigation and this led to a dichotomy result which encompasses the cases of lattices, rings, modules and quasigroups. The following theorem from [30] makes it possible to study $\mathcal{S}ysPol$ via $\mathcal{CSP}$.

Theorem 4.1. *Let $\mathbf{A}$ be a finite algebra of finite signature. Then $\mathcal{S}ysPol(\mathbf{A})$ is polynomial-time equivalent to the problem $\mathcal{CSP}(T)$ where the base set of the structure T equals that of $\mathbf{A}$ and the relations of T are the graphs of the operations of $\mathbf{A}$ and the graphs of the constants in the underlying set of $\mathbf{A}$.*

We note that it follows from a result of Klíma, Tesson and Thérien [25] that for every finite structure T, $\mathcal{CSP}(T)$ is polynomial-time equivalent to some $\mathcal{S}ysPol(\mathbf{A})$ where $\mathbf{A}$ is a semigroup. So establishing a dichotomy for $\mathcal{S}ysPol$ over the class of all finite algebras is equally hard as proving that $\mathcal{CSP}$ has a dichotomy over the class of all finite structures.

We saw in Section 1 that admitting an idempotent Maltsev condition and admitting a Taylor term operation are equivalent concepts for finite algebras. We also mentioned the following result from [9] and [28] in the introduction.

Theorem 4.2. *Let T be a finite relational structure of finite type whose clone is idempotent and contains no Taylor operation. Then $\mathcal{CSP}(T)$ is $\mathcal{NP}$-complete.*

Let $\mathbf{A}$ be a finite algebra of finite signature. By Theorem 4.1, $\mathcal{S}ysPol(\mathbf{A})$ is polynomial-time equivalent to $\mathcal{CSP}(T)$ where the relations of T are the graphs of the basic operations of $\mathbf{A}$ and the one element subsets of the base set of T. Hence the clone of T consists of the idempotent operations that preserve the graphs of the basic operations of **A**. So by Theorem 4.2, $\mathcal{S}ysPol(\mathbf{A})$ is $\mathcal{NP}$-complete if no Taylor operation preserves the graphs of the basic operations of **A**. By reformulating the preceding sentence we get to the following hardness result for $\mathcal{S}ysPol$, see [30].

Theorem 4.3. *Let* **A** *be a finite algebra of finite signature. If* **A** *has no compatible Taylor operation, then* $\mathcal{S}ysPol(\mathbf{A})$ *is* $\mathcal{NP}$*-complete.*

A dichotomy for $\mathcal{S}ysPol$ over all finite algebras yields a dichotomy for $\mathcal{CSP}$ over all finite structures and to decide the latter is considered hard. Hence when we want to prove a dichotomy theorem with respect to $\mathcal{S}ysPol$, we are compelled to make some assumption on the structure of algebras we study. In [49] we investigated $\mathcal{S}ysPol$ over the algebras that have a Taylor term operation, or equivalently admit a nontrivial idempotent Maltsev condition. This assumption on the algebras is weaker than the one we had in [30]. For example, every semilattice has a (binary) Taylor term operation but does not satisfy the requirements of the main theorem in [30].

Our strategy for proving a dichotomy theorem for $\mathcal{S}ysPol$ over finite algebras with a Taylor term operation is as follows. We assume that **A** is a finite algebra of finite signature with a Taylor term operation and investigate the specific problem in $\mathcal{CSP}$ related to $\mathcal{S}ysPol(\mathbf{A})$, described in Theorem 4.1. When doing this, by Theorem 4.3, we may consider only the case when **A** has a compatible Taylor operation. So we can restrict ourselves to the investigation of the specific problem in $\mathcal{CSP}$ related to $\mathcal{S}ysPol(\mathbf{A})$ where **A** has a Taylor term operation and a compatible Taylor operation. It turned out that algebras with this latter property have a nice structure which allows us to solve the specific problem in $\mathcal{CSP}$ in polynomial-time. Although the strategy that we sketched here is similar to the one we followed in [30], the proof of our main dichotomy theorem in [49] is quite different from that of the main theorem in [30]. The following theorem states the main result of [30].

Theorem 4.4. *Let* **A** *be a finite algebra of finite signature and* $\mathcal{V}(\mathbf{A})$ *the variety generated by* **A**. *Suppose that* **A** *omits type* 5 *and* $\mathcal{V}(\mathbf{A})$ *omits type 1. Then* $\mathcal{S}ysPol(\mathbf{A})$ *is in* $\mathcal{P}$ *if* **A** *is polynomially equivalent to a module, and* $\mathcal{S}ysPol(\mathbf{A})$ *is* $\mathcal{NP}$*-complete otherwise.*

In [49] we proved a similar but more sophisticated dichotomy theorem under the only assumption that $\mathcal{V}(\mathbf{A})$ omits type 1, i.e., **A** has a

Taylor term operation. Theorem 4.5 contains our fundamental result leading to the proof of the new dichotomy theorem. It extends the characterization of finite monoids with a compatible Taylor operation obtained in [30]. Note that this characterization did not play a role in the proof of the main theorem in [30].

A semigroup **S** is called a *semilattice of Abelian groups* if **S** has a congruence θ such that $\mathbf{S}/\theta$ is a semilattice and the blocks of θ are Abelian subgroups of **S**. The *exponent* of **S** is the least common multiple of the exponents of its Abelian subgroups, provided there exists. We note that any finite semilattice of Abelian groups **S** has a unique idempotent term operation of the form $xy^{n-1}z,\ n > 1$. Indeed, let us take two such operations corresponding to m and n, respectively, then by idempotency $x^{m+1} = x^{n+1}$ for all $x \in S$. Hence the exponent of **S** divides $m-n$, and $y^{m-1} = y^{n-1}$, i.e., the operations $xy^{m-1}z$ and $xy^{n-1}z$ coincide. The fundamental theorem of this section is as follows.

Theorem 4.5 ([49])**.** *Let M be a finite set. Let xy be a binary operation with an identity element and t a Taylor operation on M such that xy commutes with t. Then the following hold:*

1. *xy is the multiplication of a semilattice of Abelian groups.*
2. *The clone generated by t contains an idempotent ternary operation of the form $xy^{n-1}z$.*

The restriction of the following theorem to the class of monoids was proved earlier by Klíma, Tesson and Thérien in [25]. Combining their theorem with Theorem 4.3 and Theorem 4.5 we get the following result.

Theorem 4.6 ([49])**.** *Let* **M** *be a finite groupoid with an identity element. Then $\mathcal{S}ysPol(\mathbf{M})$ is in $\mathcal{P}$ if* **M** *is a semilattice of Abelian groups, and $\mathcal{S}ysPol(\mathbf{M})$ is $\mathcal{NP}$-complete otherwise.*

A *Taylor algebra* is an algebra with a Taylor term operation. A *doubly Taylor algebra* is a Taylor algebra with a compatible Taylor operation. A characterization of the structure of doubly Taylor algebras in [49] is a main ingredient in the proof of our dichotomy theorem. First we state a result on idempotent doubly Taylor algebras.

Theorem 4.7 ([49])**.** *Every finite idempotent doubly Taylor algebra is a subalgebra of a finite idempotent Taylor algebra with a compatible binary operation xy where xy is the multiplication of a semilattice of Abelian groups.*

The proof of the characterization of doubly Taylor algebras is based on the following results in tame congruence theory, see [20].

Theorem 4.8. *For any two distinct elements a and b of a finite algebra* $\mathbf{A}$ *there is a prime quotient* (α, β) *of congruences of* $\mathbf{A}$ *and a polynomial retraction* r *of* $\mathbf{A}$ *such that* $(r(a), r(b)) \in \beta \setminus \alpha$ *and* $r(\mathbf{A})$ *is an* (α, β)*-minimal set.*

Theorem 4.9. *Let* (α, β) *be a prime quotient of congruences of an algebra* $\mathbf{A}$ *where the type of* (α, β) *differs from 1. Then every* (α, β)*-minimal algebra of* $\mathbf{A}$ *has a binary basic operation with an identity element.*

Now, by using Theorem 4.7 we get to the following characterization of doubly Taylor algebras.

Theorem 4.10 ([49])**.** *A finite Taylor algebra is a doubly Taylor algebra if and only if it has a compatible idempotent ternary operation that extends to an idempotent term operation* $xy^{n-1}z$ *of a finite semilattice of Abelian groups.*

This characterization of doubly Taylor algebras made it possible to prove our dichotomy theorem. Our proof required a corollary of the next two theorems. The first theorem is a reduction theorem from [9].

Theorem 4.11. *Let* $\mathbf{A}$ *be a finite algebra such that for every finite structure* S *of finite signature whose base set coincides with that of* $\mathbf{A}$ *and whose relations are finite subpowers of* $\mathbf{A}$ *there is a polynomial-time algorithm for solving* $\mathcal{CSP}(S)$*. Then for every finite member* $\mathbf{B}$ *of the variety generated by* $\mathbf{A}$ *and every structure* T *of finite signature whose base set coincides with that of* $\mathbf{B}$ *and whose relations are finite subpowers of* $\mathbf{B}$ *there is a polynomial-time algorithm for solving* $\mathcal{CSP}(T)$*.*

In [12] Dalmau, Gavaldà, Tesson and Thérien describe a polynomial-time algorithm for solving a special type of $\mathcal{CSP}$. Their algorithm is put together from a local (so called bounded width) algorithm and an algorithm that solves $\mathcal{CSP}$ for coset structures of a group. In fact, the following theorem that we require is a special case of their Theorem 3 in [12].

Theorem 4.12. *Let* $\mathbf{M}$ *be a finite semilattice of Abelian groups, and* T *a finite relational structure of finite signature whose base set equals that of* $\mathbf{M}$*. If the idempotent term operation* $xy^{n-1}z$ *of* $\mathbf{M}$ *preserves* *the base set and the relations of T, then there exists a polynomial-time algorithm for solving $\mathcal{CSP}(T)$.*

By the previous two theorems we get the following corollary.

Theorem 4.13 ([49])**.** *Let $\mathbf{M}$ be a finite semilattice of Abelian groups and T a finite relational structure of finite signature with a base set contained in $\mathbf{M}$. If the idempotent term operation $xy^{n-1}z$ of $\mathbf{M}$ preserves the base set and the relations of T, then there exists a polynomial-time algorithm for solving $\mathcal{CSP}(T)$.*

Now, by putting together Theorems 4.3, 4.10 and 4.13 we get our main result.

Theorem 4.14 ([49])**.** *Let $\mathbf{A}$ be a finite algebra such that the variety generated by $\mathbf{A}$ omits type 1. Then $\mathcal{S}ysPol(\mathbf{A})$ is in $\mathcal{P}$ whenever $\mathbf{A}$ has a compatible idempotent ternary operation that extends to the idempotent term operation $xy^{n-1}z$ of a finite semilattice of Abelian groups, and $\mathcal{S}ysPol(\mathbf{A})$ is $\mathcal{NP}$-complete otherwise.*

We already mentioned the following result of Klíma, Tesson and Thérien in [25], although not in its precise form: for any finite structure T of finite signature there is a finite right normal band $\mathbf{B}$ such that $\mathcal{CSP}(T)$ is polynomial-time equivalent to $\mathcal{S}ysPol(\mathbf{B})$. In this respect, we note that apart from semilattices, every finite right normal band generates a variety whose type set contains type 1. Hence Theorem 4.14 says nothing about $\mathcal{S}ysPol$ over right normal bands different from semilattices. Thus, it is not possible to combine the theorem of Klíma et al. with Theorem 4.14 to prove that $\mathcal{CSP}$ has a dichotomy over all finite structures. The following theorem suggested by B. Larose generalizes Theorem 4.6 and covers some of the type 1 cases, not the case of right normal bands though.

Theorem 4.15 ([49])**.** *Let $\mathbf{A}$ be a finite algebra of finite signature that has a binary polynomial operation xy with an identity element. Then $\mathcal{S}ysPol(\mathbf{A})$ is in $\mathcal{P}$ if xy is the multiplication of a semilattice of Abelian groups and the idempotent ternary operation $xy^{n-1}z$ is a compatible operation of $\mathbf{A}$, and $\mathcal{S}ysPol(\mathbf{A})$ is $\mathcal{NP}$-complete otherwise.*

Recently, the author has found a common generalization of Theorems 4.14 and 4.15. We say that a set of transformations F of a set A is *separating* if for any two distinct elements a and b in A there exists a map $f \in F$ such that $f(a) \neq f(b)$.

Theorem 4.16. *Let* $\mathbf{A}$ *be a finite algebra of finite signature which has a separating set* F *of unary polynomial operations such that for every* $f \in F$ *there exists a binary polynomial operation* g_f *of* $\mathbf{A}$ *whose restriction to the set* $f(A)$ *is a binary operation with an identity element. Then* $\mathcal{S}ysPol(\mathbf{A})$ *is in* $\mathcal{P}$ *if* $\mathbf{A}$ *has a compatible idempotent ternary operation that extends to the idempotent term operation* $xy^{n-1}z$ *of a finite semilattice of Abelian groups, and* $\mathcal{S}ysPol(\mathbf{A})$ *is* $\mathcal{NP}$*-complete otherwise.*

Proof. Suppose first that $\mathbf{A}$ has a compatible Taylor operation t. Let A denote the underlying set of $\mathbf{A}$. Clearly, the operations of F are all endomorphisms of the algebra (A, t) and for all $f \in F$ the $(f(A), t|_{f(A)})$ are subalgebras of (A, t). Since F is separating, (A, t) embeds into the direct product $\mathbf{B}$ of the $(f(A), t|_{f(A)})$, $f \in F$. The operation g acting componentwise as g_f, $f \in F$, on $\mathbf{B}$ is a compatible binary operation of $\mathbf{B}$ which has an identity element. By invoking Theorem 4.5, $xy = g(x, y)$ is a multiplication of a semilattice of Abelian groups and $xy^{n-1}z$ is in the clone generated by $t_{\mathbf{B}}$. So there is a finite semilattice of Abelian groups on the base set of $\mathbf{B}$ whose idempotent term operation $xy^{n-1}z$ restricts to a copy of $\mathbf{A}$ as a compatible idempotent operation. Then by Theorem 4.13, $\mathcal{S}ysPol(\mathbf{A})$ is in $\mathcal{P}$.

If $\mathbf{A}$ has no compatible Taylor operation, then by Theorem 4.3, $\mathcal{S}ysPol(\mathbf{A})$ is $\mathcal{NP}$-complete. □

By Theorems 4.8 and 4.9 the conditions of the preceding theorem are satisfied for any finite algebra of finite signature that omits type 1. So we get the following generalization of Theorem 4.14.

Corollary 4.17. *Let* $\mathbf{A}$ *be a finite algebra of finite signature that omits type 1. Then* $\mathcal{S}ysPol(\mathbf{A})$ *is in* $\mathcal{P}$ *if* $\mathbf{A}$ *has a compatible idempotent ternary operation that extends to the idempotent term operation* $xy^{n-1}z$ *of a finite semilattice of Abelian groups, and* $\mathcal{S}ysPol(\mathbf{A})$ *is* $\mathcal{NP}$*-complete otherwise.*

Note that the above generalization is proper since there are finite algebras $\mathbf{A}$ omitting type 1 such that the variety generated by $\mathbf{A}$ admits type 1, see [20] for examples.

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Appendix

containing the copies of papers:

[29] B. Larose, L. Zádori, Finite posets and topological spaces in locally finite varieties, Algebra Universalis, **52** (2-3), 119-136, 2004.

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