TYPICAL FACES OF BEST APPROXIMATING POLYTOPES WITH A RESTRICTED NUMBER OF EDGES

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Abstract. Let K be a convex body in $\mathbb{E}^3$ with a C^2 smooth boundary. In this article, we investigate polytopes with at most n edges circumscribed about K or inscribed in K, which approximate K best in the Hausdorff metric. The asymptotic behaviour of the distance, as a function of n, of such best approximating polytopes and K is known, see [3] for an asymptotic formula. In this article, we prove that the typical faces of the best approximating circumscribed or inscribed polytopes in the Hausdorff metric with at most n edges are asymptotically squares with respect to the second fundamental form of ∂K.

1. Notation and conventions

We shall work in d-dimensional Euclidean space $\mathbb{E}^d$, with origin o, and scalar product $\langle \cdot, \cdot \rangle$, and induced norm $|\cdot|$. We shall not distinguish between the Euclidean space and the underlying vector space, and we will use the words *point* and *vector* interchangeably, as we need them. Points of $\mathbb{E}^d$ are denoted by small-case letters of the roman alphabet, and sets by capitals. For reals we use either Greek letters or small-case letters. For a compact convex set K, we write aff K for its affine hull, and relint K for its relative interior. A compact convex set K with nonempty interior is called a convex body. If the dimension of K is two, then we call it a *convex disc*. For the sake of brevity, we shall use the term *unit disc* for the unit radius circular disc. B^d stands for the unit ball in $\mathbb{E}^d$ centred at the origin. Volume in $\mathbb{E}^d$ is denoted by $V(\cdot)$, and two-dimensional Hausdorff measure is denoted by $A(\cdot)$. If A, B are subsets of $\mathbb{E}^d$, then the *convex hull* of A and B is denoted by $[A, B]$.

There are numerous ways to define metrics on the space of convex bodies $\mathcal{K}^d$, of which the Hausdorff metric is one of the most natural and applicable ones. For $K, L \in \mathcal{K}^d$ the *Hausdorff distance* is defined by

$$\delta_H(K, L) = \min\{\lambda \geq 0 \mid K \subset L + \lambda B^d, L \subset K + \lambda B^d\}.$$

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Then δ_H is a metric on $\mathcal{K}^d$, called the *Hausdorff metric*. For further details on convex sets and related measures consult the monographs of R. Schneider [14] and P.M. Gruber [10].

Let K be a convex body in $\mathbb{E}^d$, and let ∂K denote its boundary. We always integrate on ∂K with respect to the $(d-1)$-dimensional Hausdorff measure. We say that K has C^2 boundary if for any $x \in \partial K$, a neighbourhood of x in ∂K is the graph of a convex C^2 function f that is defined in the orthogonal projection of that neighbourhood into the tangent plane T_x at x. For $x \in \partial K$ we write Q_x to denote the second fundamental form at x that is, the quadratic form representing the second derivative of f at x. Because of the convexity of K, Q_x is positive semi-definite. Its eigenvalues are the principal curvatures, and its determinant $\kappa(x)$ is the *Gauss-Kronecker curvature* of ∂K at x. Clearly, $\kappa(x) \geq 0$ for all $x \in \partial K$. If, in addition, $\kappa(x) > 0$ for all $x \in \partial K$, then we say that the boundary of K is C^2_+.

Throughout the paper we shall use the customary notation for the magnitude of functions. Let f and g be functions of positive integers. We write $f(n) = O(g(n))$ if there exists a constant c depending on the given convex body K such that $|f(n)| \leq c \cdot g(n)$ for all $n \geq 1$, and $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$. Furthermore, we write $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$.

2. History

The starting point of polytopal approximation was the work of L. Fejes Tóth. He investigated some basic questions in the plane and in the 3-dimensional space. The first results on this topic in higher dimensions are due to R. Schneider [13].

Let K be a convex body in the d-dimensional Euclidean space with a sufficiently smooth boundary, and let $0 \leq k \leq d-1$ be an integer. We define the set $\mathcal{P}_n := \{P | P \text{ is a polytope with at most } n \ k\text{-faces}\}$. Then there exists a (not necessarily unique) polytope $P_n \in \mathcal{P}_n$ such that $\delta(K, P_n) = \inf\{\delta(P, K) | P \in \mathcal{P}_n\}$, where δ stands for any metric defined on the space of convex bodies. Note that it is customery to make further restrictions on $\mathcal{P}_n$, for example we can consider polytopes which contain K (circumscribed) or which are contained in K (inscribed). We are usually interested in describing various properties of P_n which we do by formulating asymptotic results as $n \to \infty$.

Most of earlier papers deal with the cases when $k = 0$ or $k = d-1$, that is, the number of vertices or facets is restricted. There are asymptotic formulas on the distance between K and P_n in every dimension for most well-known metrics, such as the volume of the symmetric difference, the Hausdorff metric, the Banach-Mazur metric, the L_1 metric and the Schneider distance, although not all the constans are known. These asymptotic formulas are due to R. Schneider, P. M. Gruber, S. Glasauer, M. Ludwig and K. J. Böröczky.

In this paper, we determine the typical faces of a special class of best approximating polytopes using a recent asymptotic formula of [3]. (The exact meaning of "typical faces" will be explained later.)

We shall investigate the case when $d = 3$ and $k = 1$ and we shall use Hausdorff metric to measure the distance between bodies. For completeness, we repeat the definitions in this special case. Let $\mathcal{P}_n^c(K)$ denote the set of polytopes circumscribed about K and having at most n edges. There exists a polytope P_n^c, not unique in general, such that

$$\delta_H(K, P_n^c) = \inf\{\delta_H(K, P) : P \in \mathcal{P}_n^c(K)\}.$$

We may similarly define the polytope P_n^i using inscribed polytopes instead of circumscribed. The following theorem was the main result of [3].

Theorem 1. [3] *If $K \in \mathcal{K}^3$ is a convex body with C^2 boundary, then*

$$\delta_H(K, P_n^c), \delta_H(K, P_n^i) \sim \frac{1}{2}\int_{\partial K} \kappa^{1/2}(x)dx \cdot \frac{1}{n} \quad \textit{as } n \to \infty. \tag{1}$$

3. Main Result

If C and D are two convex discs, then we say that C is ε-close to D if there exist $x \in C$ and $y \in D$ such that

$$(1+\varepsilon)^{-1} \cdot (C - x) \subset D - y \subset (1+\varepsilon) \cdot (C - x).$$

Let Q be a positive definite, non-degenerate quadratic form on the Euclidean plane with the positive definite, symmetric matrix C. It is well known that C can be written in the form $C = A^T A$, with a non-singular matrix A (this representation is not unique). Denote φ_A the linear transformation defined by A. A polygon P is called regular with respect to Q, if $\varphi_A(P)$ is a regular polygon (in Euclidean sense). Furthermore, if $\varphi_A(P)$ is of area 1, then we say that P is a regular unit polygon with respect to Q. In particular, we will refer to a regular quadrilateral with respect to Q as a Q square. One can see that the definition does not depend on the choice of A.

Now we are going to define what "typical faces with respect to a density function" mean. We remark that the same definition was given in [5]. Let $\rho : \partial K \to \mathbb{R}$ be a non-negative function such that if $\kappa(x) > 0$ then $\rho(x) > 0$. Let R_n be a sequence of polytopes and let $f(n)$ denote the number of facets of R_n. Suppose that $o \in \operatorname{int} R_n$ and $f(n) \to \infty$ if $n \to \infty$. Furthermore, suppose that there exists a positive zero-sequence $\nu(n)$ with the following property. For all but $\nu(n)$ percent of the facets F of R_n we have that F is a k-gon, there is a unique point $x_F \in \partial K$ such that $u(x_F)$ is an exterior normal also to F, Q_{x_F} is positive definite, and F is $\nu(n)$-close to a k-gon which is regular with respect to Q_{x_F} and is of area

$$\frac{\int_{\partial K} \rho(x)dx}{f(n)\rho(x_F)}.$$

In this case we say that the typical facets of R_n are asymptotically regular k-gons with respect to the density function ρ. Note that the density function ρ only determines the *area* of the facets approximately, whereas their shape only depends on K.

Let $\mathcal{P}_n^c(K)$ denote the set of polytopes circumscribed about K and having at most n edges. There exists a polytope P_n^c, not unique in general, such that

$$\delta_H(K, P_n^c) = \inf\{\delta_H(K, P) : P \in \mathcal{P}_n^c(K)\}.$$

We may similarly define the polytope P_n^i using inscribed polytopes instead of circumscribed.

The main result of this article is the following theorem.

Theorem. *Let $K \subset \mathbb{E}^3$ be a convex body with C^2 smooth boundary. The typical facets of P_n^c and P_n^i are squares with respect to the density function $\kappa^{1/2}(x)$ as $n \to \infty$.*

Below, we briefly review what is known of the shape of typical faces of best approximating polytopes. P. M. Gruber, in [8], investigated, for sufficiently smooth convex bodies in three-dimensional Euclidean space, the typical shapes of the facets of best approximating inscribed polytopes with at most n vertices as $n \to \infty$, and of circumscribed polytopes with at most n facets as $n \to \infty$. He considered best approximating polytopes according to the Hausdorff distance, Banach-Mazur distance and Schneider's notion of distance. He proved for these three metrics that in the inscribed case the typical shape of the facets are asymptotically close to regular triangles in an appropriate sense, and in the case of circumscribed polytopes they are asymptotically close to regular hexagons in a suitable sense depending on the metric. In [9], P. M. Gruber proved that for a convex body with C^2 boundary and everywhere positive Gauss-Kronecker curvature, the typical faces of a minimal volume circumscribed polytope with n facets are asymptotically close to regular hexagons in a suitable sense as $n \to \infty$. This work was continued by K. J. Böröczky, P. Tick and G. Wintsche in [5], where they extended Gruber's result for convex bodies with no restriction on the Gauss-Kronecker curvature. They also showed that the typical faces of maximal volume inscribed polytopes with at most n vertices are asymptotically close to regular triangles in a suitable sense as $n \to \infty$.

4. Lemmas and tools

In this section we recall some results and notation from [3].

Let X and X' be relatively open, Jordan-measurable subsets of ∂K with the following properties.

$$\text{cl } X \subset \text{relint } X' \subset \partial K, \tag{2}$$

$$\exists \eta > 0 : \text{ all principal curvatures at } x \in \text{cl } X' \text{ are at least } \eta. \tag{3}$$

Let C be a convex polygon tangent to K at $x \in \operatorname{relint} C$ such that the orthogonal projection of $\operatorname{int} K$ into $\operatorname{aff} C$ covers C. For any function $f : C \to \mathbb{R}$, we define its graph

$$\Gamma(f) = \{z - f(z)u(x) : z \in C\}.$$

Let $f_C : C \to \mathbb{R}$ denote the convex C^2 function with $\Gamma(f_C) \subset \partial K$. We shall use $p_C : \mathbb{E}^3 \to \operatorname{aff} C$ to denote the orthogonal projection onto $\operatorname{aff} C$, and $\Pi_{\partial K} : C \to \partial K$ for the nearest point map onto ∂K.

The following lemma is identical to Lemma 2 in [3].

Lemma 1. [3] *For each $\varepsilon > 0$, there exists a $\delta(K, \varepsilon) = \delta > 0$ such that if $C \subset x + \delta B^3$ is a convex polygon touching K at $x \in X \cap \operatorname{relint} C$, then the following statements hold. We have $\Gamma(f_C) \subset X'$, and*

$$\text{for all } y \in C,\ (1+\varepsilon^3)^{-1} Q_x \leq q_y \leq (1+\varepsilon^3) Q_x, \tag{4}$$

$$\text{for all } z \in \Gamma(f_C),\ \langle u(z), u(x)\rangle \geqq (1+\varepsilon)^{-1} \tag{5}$$

$$p_C(\Pi_{\partial K}(C)) \supseteq (1-\varepsilon)(C-x) + x. \tag{6}$$

The following lemma will be one of the most important tools in the proof the Theorem. Inequality (7) was proved in [3]. This extended version establishes the stability of the original inequality.

Lemma 2. *Let $q(x)$ be a positive definite quadratic form on $\mathbb{R}^2$ and $\alpha \leq 0$ a real number. Let $G = [p_1, p_2, \ldots, p_k]$ be a k-gon with vertices $\{p_i\}$. Then*

$$\max_{x \in G} (q(x) - \alpha) \geqq \frac{2}{k} \cdot A(G)\sqrt{\det q}. \tag{7}$$

Furthermore, if $k \neq 4$, then

$$\max_{x \in G}(q(x) - \alpha) > 1.04 \cdot \frac{2}{k} \cdot A(G)\sqrt{\det q}. \tag{8}$$

If

$$\max_{x \in G}(q(x) - \alpha) \leq (1 + \varepsilon) \cdot \frac{2}{k} \cdot A(G)\sqrt{\det q}, \tag{9}$$

then G is $O(\sqrt[4]{\varepsilon})$-close to a q-square.

Proof. Inequality (7) was proved in [3].

Now, we are going to prove inequality (8). We may suppose that $q(x) = x^2$ and $\alpha = 0$. Since x^2 is a convex function we have that

$$\max_{x \in G} x^2 = \max_{x \in \{p_i\}} x^2.$$

Without loss of generality, we may suppose that this maximum is attained at p_1. From this it follows that G is contained in a circle of radius R centered at the origin, where $R = d(o, p_1)$. It is well-known that the regular k-gon has the maximal area among k-gons inscribed in a circle. This implies that

$$A(G) \leq \frac{R^2 \cdot \sin\frac{2\pi}{k} \cdot k}{2},$$

which proves inequality (7). Furthermore, if $k \neq 4$ then $\sin(2\pi/k) \leq 0.96$ and from this follows inequality (8).

Now, we turn to the proof of inequality (9). If $o \notin G$ then apply the statement to $G - p$ where p is the nearest point of G to the origin. So we may suppose that G is a quadrilateral and that $o \in G$. We may further suppose that p_1, p_2, p_3, p_4 are in positive order and $|p_1|$ is maximal. Let us denote the angles between p_i and p_{i+1} with φ_i ($p_5 \equiv p_1$).

With these notation, we have that

$$\max_{x \in G} x^2 = {p_1}^2.$$

On the one hand,

$$A(G) = \frac{\sum_1^4 |p_i||p_{i+1}| \sin\varphi_i}{2} \leq {p_1}^2 \cdot \frac{\sum_1^4 \sin\varphi_i}{2},$$

from the assumptions of the lemma we obtain that

$$(1 + \varepsilon)^{-1} \leq \frac{\sum_1^4 \sin\varphi_i}{4}.$$

This readily implies that $|\varphi_i - \pi/2| \leq O(\sqrt{\varepsilon})$ holds for all i.

On the other hand,

$$A(G) = \frac{\sum_1^4 |p_i||p_{i+1}| \sin\varphi_i}{2} \le \frac{\sum_1^4 |p_i||p_{i+1}|}{2},$$

from which we have that

$$4p_1^2 \le (1+\varepsilon) \sum_1^4 |p_i||p_{i+1}|,$$

and finally that

$$|p_i| \ge (1+\varepsilon)^{-1}|p_1|.$$

From the above observations inequality (9) follows by elementary calculations. □

We will apply the previous lemma to prove the circumscribed case. We need a slight modification of it to obtain the inscribed case.

Lemma 3. *Let $q(x)$ be a positive definite quadratic form on $\mathbb{R}^2$. Let $G = [p_1, p_2, \dots, p_k]$ be a k-gon with vertexes $\{p_i\}$ and α a real number such that $\alpha \ge q(x)$ for all $x \in G$. Then*

$$\max_{x\in G}(\alpha - q(x)) \geqq \frac{2}{k} \cdot A(G)\sqrt{\det q}.$$

Furthermore if $k \ne 4$ then

$$\max_{x\in G}(\alpha - q(x)) > 1.04 \cdot \frac{2}{k} \cdot A(G)\sqrt{\det q}.$$

If

$$\max_{x\in G}(\alpha - q(x)) \le (1+\varepsilon) \cdot \frac{2}{k} \cdot A(G)\sqrt{\det q},$$

then G is $O(\sqrt[4]{\varepsilon})$ close to a q-square.

Proof. The first part of the lemma was proved in Lemma 3. in the fourth section in [3].

We would like to use Lemma 2. Suppose that $o \in G$. Since $q(o) = 0$ we obtain that $\max_{x\in G}(\alpha - q(x)) = \alpha \ge \max_{x\in G} q(x)$. We can apply Lemma 2. and every statement of the lemma readily follows.

Now suppose that $o \notin G$ and $q(x) = x^2$. Here we can repeat a part of the proof from [3].

First suppose that $k \geqq 4$. Let p be the nearest point of G to the origin, and $d = |op|$. Then the circle of radius $\sqrt{\alpha - d^2}$ centred at p contains G. Furthermore, there exists a line through p which separates G from the origin. We conclude that

$$\max_{x\in G}(\alpha - q(x)) = \alpha - d^2 \ge 1.04 \cdot \frac{2}{k} \cdot \frac{\pi}{2}(\alpha - d^2) \geqq 1.04 \cdot \frac{2}{k} \cdot A(G).$$

The last case to be checked is when G is a triangle and $o \notin G$. Using the same notation as before, we obtain

$$\max_{x\in G}(\alpha - q(x)) = \alpha - d^2 \geq A(G) > 1.04 \cdot \frac{2}{3}A(G).$$

$\square$

Lemma 4. *Let $a_i, b_i \in \mathbb{R}_+$ for $i = 1, 2, \ldots, m$. If there exists a λ such that $a_i/b_i \leq \lambda$ holds for all $i = 1, 2, \ldots, m$ then $(\sum_{i=1}^m a_i)/(\sum_{i=1}^m b_i) \leq \lambda$.*

Proof. If we sum the inequalities $a_i \leqq \lambda b_i$, then we get that $\sum a_i \leqq \sum \lambda b_i$. Dividing both side with $\sum b_i$ we obtain the statement of the lemma.

$\square$

Now, we are going to introduce the notation and conventions which will be used to state Lemma 5. We use δ and X from Lemma 1. For $x \in X$, let $C \subset x + \delta B^3$ be a convex polygon such that C touches K at $x \in \operatorname{relint} C$. Furthermore, let C' be a convex polygon such that $C' \subset \operatorname{relint} C$ and $C \subset p_C(K)$. We write f to denote the convex function on C such that $\Gamma(f) \subset \partial K$. We shall use l_y for the linear form representing the first derivative of f at $y \in C$, and, as usual, q_y to denote the quadratic form representing the second derivative of f at $y \in C$. Note that $Q_x = q_x$.

Next, let P be a polytope with $C \subset p_C(P)$, and let $F_1, \ldots, F_k$ be the faces of P with the following properties. For $i = 1, \ldots, k$, the exterior unit normal of F_i encloses an acute angle with $u(x)$ and $p_C(F_i) \cap C' \neq \emptyset$. Furthermore, we assume that

$$p_C(F_i) \subset C,\ i = 1, \ldots, k,$$

and for any F_i, there exists an $a_i \in C$ such that the exterior unit normal to F_i coincides with the exterior unit normal to the graph of f at $a_i - f(a_i)u(x)$. In particular, aff F_i is the graph of the function

$$\varphi_i(y) = f(a_i) + l_{a_i}(y - a_i) + \alpha_i$$

for some $\alpha_i \in \mathbb{R}$. Finally, we define $\Pi_i = p_C(F_i)$.

Lemma 5 (Transfer lemma)**.** [3] *Let $\varepsilon \in (0, 1/4)$. Using the notation as above, we assume that Q_x is positive definite and for all $z \in C$, we have $(1+\varepsilon)^{-1}Q_x \leqq q_z \leqq (1+\varepsilon)Q_x$ and $\langle u(x), u(w)\rangle \geq (1+\varepsilon)^{-1}$ for $w = z - f(z)u(x)$. Then the following statements hold.*

(i) *If $K \subset P$ then each $\alpha_i \leq 0$, and*

$$\delta_H(P, K) \geqq (1 - 2\varepsilon) \max_{i=1,\ldots,k} \max_{y\in\Pi_i} \left(\tfrac{1}{2} Q_x(y - a_i) - \alpha_i\right).$$

(ii) *If $P \subset K$ then each $\alpha_i > 0$. Letting $\alpha'_i = (1+\varepsilon)\alpha_i$ for $i = 1,\dots,k$, we have $\frac{1}{2}Q_x(y-a_i) \le \alpha'_i$ for $y \in \Pi_i$, $i = 1,\dots,k$, and*

$$\delta_H(P,K) \geqq (1-4\varepsilon) \max_{i=1,\dots,k} \max_{y\in\Pi_i} \left(\alpha'_i - \tfrac{1}{2}Q_x(y-a_i)\right).$$

The next three statements contain simple information which will be used throughout the article. Since their proofs are elementary we leave them to the reader.

Proposition 1. *If C, D and K are convex disks and $0 < \varepsilon < 1$ is a positive real number such that C is ε-close to D and D is ε-close to K then C is 3ε-close to K.*

Proposition 2. *Let Σ_1 and Σ_2 be 2-dimensional planes in $\mathbb{E}^3$ such that their angle is smaller than ε, and suppose that D is a convex disc in Σ_1. Let ρ denote the rotation which takees Σ_2 into Σ_1. Then $\rho(p_{\Sigma_2}(D))$ is ε-close to D, where p_{Σ_2} denotes the orthogonal projection onto Σ_2.*

Proposition 3. *Let q_1 and q_2 be two positive definite quadratic forms on the Euclidean plane such that*

$$(1+\varepsilon^2)^{-1}q_1(x) \le q_2(x) \le (1+\varepsilon^2)q_1(x).$$

Then there exists a unit square of q_1 which is ε-close to a unit square of q_2.

5. Proof of the Theorem

We are going to prove the Theorem only for best approximating circumscribed polytopes. The proof for inscribed polytopes is very similar.

From (1) it follows that there exists a positive sequence b_n with $\lim b_n = 0$ such that

$$\delta_H(K, P_n^c) \le (1+b_n)\cdot \frac{1}{2}\int_{\partial K} \kappa^{1/2}(x)dx \cdot \frac{1}{n}. \tag{10}$$

Let $\varepsilon > 0$ be fixed. We will divide ∂K into two parts: the "flat" part and the "curved" part in the following way. Let X, $X' \subset \partial K$ be defined in such a way that they satisfy the conditions (2) and (3), furthermore, that

$$\int_X \kappa^{1/2}(x)dx > (1-\varepsilon)\int_{\partial K} \kappa^{1/2}(x)dx. \tag{11}$$

X can naturally be called the "curved" part of ∂K, and its complement is the "flat" part.

We are going to construct a "large" auxiliary polytope $M = M(\varepsilon)$ circumscribed about K. We require that M has the following property. If C is a facet of M and $\Pi_{\partial K} C \subset X'$, then diam $C < \delta$, where δ is the same as in Lemma 1. Let $\widehat{\mathcal{C}}$ be the family of all facets C of M such that $\Pi_{\partial K} C \cap X \neq \emptyset$. For all $C \in \widehat{\mathcal{C}}$, there is a unique $x_C \in X'$ such that $u(x_C)$ is normal to C. Define

$$\mathcal{C} = \{x_C + (1-2\varepsilon)(C - x_C) \mid C \in \widehat{\mathcal{C}},\ \Pi_{\partial K}(x_C + (1-2\varepsilon)(C - x_C)) \cap X \neq \emptyset\}.$$

The properties of X and Lemma 1 yield

Lemma 6.

$$\sum_{C \in \mathcal{C}} \kappa^{1/2}(x_C) \cdot A(C) \geq (1 - O(\varepsilon)) \int_{\partial K} \kappa^{1/2}(x) dx.$$

One can see that there exists an $\omega = \omega(K, \varepsilon) > 0$ such that if F is a face of P_n^c and $\exists C \in \widehat{\mathcal{C}}$ with $\Pi_{\partial K}(C) \cap \Pi_{\partial K}(F) \neq \emptyset$ then diam $F \leqq \omega/\sqrt{n}$. Now, if $n > N_\varepsilon$ then for every facet F of P_n^c there is at most one $C \in \mathcal{C}$ such that $\Pi_{\partial K}(F) \cap \Pi_{\partial K}(C) \neq \emptyset$. Let us denote by $\mathcal{C}_C$ the set of those facets of P_n^c which are "above" C; namely, their projection into aff$C$ intersects C, and whose exterior unit normal encloses an acute angle with $u(x_C)$. For any $F \in \mathcal{C}_C$, let $\Pi_F = p_C(F)$, let $x_F \in \partial K$ satisfy that $u(x_F)$ is the exterior unit normal to F. Let k_F denote the number of sides of F. Now if n is large, then $a_F = p_C(x_F)$ lies in C, and let α_F be defined by $x_F - \alpha_F u(x_C) \in \text{aff} F$, hence $\alpha_F \leq 0$. Let us denote with $\mathcal{F}$ the set of facets of P_n^c, furthermore let $\mathcal{F}_+ := \{F \in \mathcal{F} : \Pi_{\partial K} F \cap X \neq \emptyset\}$ and $\mathcal{F}_0 := \mathcal{F} \setminus \mathcal{F}_+$.

Our first goal is to show that there are only very few facets over the flat part compared to the curved part of ∂K. To obtain the following sequence of inequalities, apply Lemma 5 and Lemma 2.

$$\begin{aligned}
\delta_H(P_n^c, K) &\geq (1 - O(\varepsilon)) \max_{C \in \widehat{\mathcal{C}}} \max_{F \in \mathcal{C}_C} \max_{y \in \Pi_F} \left(\tfrac{1}{2} Q_{x_C}(y - a_F) - \alpha_F\right) \\
&\geq (1 - O(\varepsilon)) \cdot \max_{C \in \widehat{\mathcal{C}}} \max_{F \in \mathcal{C}_C} \frac{\kappa^{1/2}(x_C) \cdot A(\Pi_F)}{k_F} \qquad (12) \\
&\geq (1 - O(\varepsilon)) \cdot \left(\max_{C \in \widehat{\mathcal{C}}} \max_{F \in \mathcal{C}_C \cap \mathcal{F}_+} \frac{\kappa^{1/2}(x_C) \cdot A(\Pi_F)}{k_F} \right).
\end{aligned}$$

Now, using Lemma 4 and Lemma 6 we get

$$\max_{C\in\widehat{\mathcal{C}}}\max_{F\in\mathcal{C}_C\cap F_+}\frac{\kappa^{1/2}(x_C)\cdot A(\Pi_F)}{k_F}\geq$$
$$\geq(1-O(\varepsilon))\cdot\frac{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C\cap\mathcal{F}_+}(\kappa^{1/2}(x_C)\cdot A(\Pi_F))}{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C\cap\mathcal{F}_+}k_F}\geq$$
$$\geq(1-O(\varepsilon))\int_{\partial K}\kappa^{1/2}(x)dx\cdot\frac{1}{2n_+},$$

where n_+ stands for number of edges in $\mathcal{F}_+$. From (10) it follows that if $b_n<\varepsilon$, then $n_+>(1-O(\varepsilon))n$.

We note that for all elements $F\in\mathcal{F}_+$, there is a unique point $x_F\in\partial K$ such that $u(x_F)$ is an exterior normal also to F, and Q_{x_F} is positive definite.

In the next step, we use the same inequalities in a different way.

$$\max_{C\in\widehat{\mathcal{C}}}\max_{F\in\mathcal{C}_C\cap F_+}\frac{\kappa^{1/2}(x_C)\cdot A(\Pi_F)}{k_F}\geq(1-O(\varepsilon))\int_{\partial K}\kappa^{1/2}(x)dx\cdot\frac{1}{2n_+}\geq$$
$$\geq(1-O(\varepsilon))\max_{C\in\widehat{\mathcal{C}}}\max_{F\in\mathcal{C}_C\cap\mathcal{F}_+}\max_{y\in\Pi_F}\left(\tfrac{1}{2}\,Q_{x_C}(y-a_F)-\alpha_F\right)\tag{13}$$

holds if n is large enough.

Next, we divide the facets in $\mathcal{F}_+$ into two classes. Let $\mathcal{F}_{nsq}\subseteq\mathcal{F}_+$ contain those elements $F\in\mathcal{F}_+$ for which

$$\max_{y\in\Pi_F}(\tfrac{1}{2}\,Q_{x_C}(y-a_F)-\alpha_F)\geq(1+\sqrt[3]{\varepsilon})\cdot\frac{1}{k_F}\cdot A(\Pi_F)\kappa^{1/2}(x_C),$$

and let $\mathcal{F}_{sq}=\mathcal{F}_+\setminus\mathcal{F}_{nsq}$, the other class. Now, we continue the sequence of inequalities in (13). In particular,

$$\max_{C\in\widehat{\mathcal{C}}}\max_{F\in\mathcal{C}_C\cap\mathcal{F}_+}\max_{y\in\Pi_F}(\tfrac{1}{2}\,Q_{x_C}(y-a_F)-\alpha_F)\geq$$
$$\geq\max(\max_{C\in\widehat{\mathcal{C}}}\max_{F\in\mathcal{C}_C\cap\mathcal{F}_{sq}}\frac{\kappa^{1/2}(x_C)\cdot A(\Pi_F)}{k_F},$$
$$,\max_{C\in\widehat{\mathcal{C}}}\max_{F\in\mathcal{C}_C\cap\mathcal{F}_{nsq}}(1+\sqrt[3]{\varepsilon})\cdot\frac{\kappa^{1/2}(x_C)\cdot A(\Pi_F)}{k_F})$$

Now, we apply Lemma 4 two times and we obtain that

$$\max_{C\in\widehat{\mathcal{C}}}\max_{F\in\mathcal{C}_C\cap F_+}\frac{\kappa^{1/2}(x_C)\cdot A(\Pi_F)}{k_F}\geq$$
$$\geq(1-O(\varepsilon))\max\Big(\frac{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C\cap\mathcal{F}_{sq}}(\kappa^{1/2}(x_C)\cdot A(\Pi_F))}{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C\cap\mathcal{F}_{sq}}k_F},$$
$$,\frac{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C\cap\mathcal{F}_{nsq}}(\kappa^{1/2}(x_C)\cdot A(\Pi_F))}{(1-\sqrt[4]{\varepsilon})\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C\cap\mathcal{F}_{nsq}}k_F}\Big)$$

In order to reach a contradiction, assume that

$$2n_{nsq}:=\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C\cap\mathcal{F}_{nsq}}k_F>2n_+\sqrt[4]{\varepsilon}.$$

Using Lemma 4 again, it follows that

$$\max_{C\in\widehat{\mathcal{C}}}\max_{F\in\mathcal{C}_C\cap F_+}\frac{\kappa^{1/2}(x_C)\cdot A(\Pi_F)}{k_F}\geq$$
$$(1-O(\varepsilon))\frac{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C\cap\mathcal{F}_+}(\kappa^{1/2}(x_C)\cdot A(\Pi_F))}{2n_{sq}+(1-\sqrt[4]{\varepsilon})2n_{nsq}}$$
$$\geq(1+\sqrt[3]{\varepsilon})(1-O(\varepsilon))\frac{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{F}_+}A(\Pi_F)}{2n_+}$$

This is clearly a contradiction, since $(1+\sqrt[3]{\varepsilon})(1-O(\varepsilon))>1$ if ε is small enough. Thus, we have obtained, that $n_{nsq}<\sqrt[4]{\varepsilon}n_+$, which implies that $|\mathcal{F}_{nsq}|<\sqrt[4]{\varepsilon}|\mathcal{F}_+|$.

By the definition of $\mathcal{F}_{sq}$, we know that for all $F\in\mathcal{F}_{sq}$

$$\max_{y\in\Pi_F}(\tfrac{1}{2}\,Q_{x_C}(y-a_F)-\alpha_F)\leq(1+\sqrt[3]{\varepsilon})\cdot\frac{1}{k_F}\cdot A(\Pi_F)\kappa^{1/2}(x_C).$$

The conditions of statement (9) in Lemma 2 are satisfied, and so all Π_F for $F\in\mathcal{F}_{sq}$ are $\sqrt[16]{\varepsilon}$-close to a Q_{x_C}-square. Furthermore, if ε is small enough, then from inequality (8) in Lemma 2 it follows that $\mathcal{F}_{sq}$ contains only quadrilaterals, or equivalently, for all $F\in\mathcal{F}_{sq}$ we have that $k_F=4$.

Next we show that each $F\in\mathcal{F}_{sq}$ is close to a Q_{x_F}-square. To obtain this, we recall (4) and (5) from Lemma 1 and Propositions 1, 2 and 3. From these the claim readily follows.

Finally, we will consider the areas. We proved that almost every face belongs to $\mathcal{F}_{sq}$. Now, we will decompose $\mathcal{F}_{sq}$ into two sets. To simplify the notation, we introduce

$$T=\max_{C\in\widehat{\mathcal{C}}}\max_{F\in\mathcal{C}_C\cap\mathcal{F}_{sq}}\kappa^{1/2}(x_C)A(\Pi_F).$$

And now let

$$\mathcal{F}_M = \{F \in \mathcal{F}_{sq} : \kappa^{1/2}(x_C)A(\Pi_F) > (1-\sqrt[4]{\varepsilon})T, F \in \mathcal{C}_C\}$$

and $\mathcal{F}_N = \mathcal{F}_{sq} \backslash \mathcal{F}_M$. On the contrary, assume $|\mathcal{F}_N| > (1-\sqrt[4]{\varepsilon})|\mathcal{F}_{sq}|$.

On one hand observe that

$$\frac{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C\cap\mathcal{F}_{sq}}(\kappa^{1/2}(x_C)\cdot A(\Pi_F))}{|\mathcal{F}_{sq}|} \leqq \frac{|\mathcal{F}_M|T + |\mathcal{F}_N|(1-\sqrt[4]{\varepsilon})T}{|\mathcal{F}_{sq}|} =$$

$$= T - \frac{|\mathcal{F}_N|}{|\mathcal{F}_{sq}|}T\sqrt[4]{\varepsilon} \leqq (1-\sqrt{\varepsilon})T.$$

On the other hand

$$\frac{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C\cap\mathcal{F}_{sq}}(\kappa^{1/2}(x_C)\cdot A(\Pi_F))}{4|\mathcal{F}_{sq}|} =$$

$$= \frac{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C\cap\mathcal{F}_{sq}}(\kappa^{1/2}(x_C)\cdot A(\Pi_F))}{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C\cap\mathcal{F}_{sq}}k_F} \geqq$$

$$\geqq (1-O(\varepsilon))\int_{\partial K}\kappa^{1/2}(x)dx\cdot\frac{1}{2n} \geqq$$

$$\geqq (1-O(\varepsilon))\max_{C\in\widehat{\mathcal{C}}}\max_{F\in\mathcal{C}_C\cap\mathcal{F}_{sq}}\frac{\kappa^{1/2}(x_C)\cdot A(\Pi_F)}{k_F} = \frac{(1-O(\varepsilon))}{4}T.$$

This clearly leads to a contradiction if ε small enough. From the estimates above, we have that

$$\frac{T}{4} \geqq (1-O(\varepsilon))\int_{\partial K}\kappa^{1/2}(x)dx\cdot\frac{1}{2n} \geqq \frac{(1-O(\varepsilon))}{4}T.$$

It follows from Lemma 1 that we may estimate $\kappa(x_F)$ by $\kappa(x_C)$ and $A(F)$ by $A(\Pi_F)$, and so if $F \in \mathcal{F}_M$, then

$$(1+O(\sqrt[4]{\varepsilon}))\cdot\frac{2\int_{\partial K}\kappa^{1/2}(x)dx}{n\kappa^{1/2}(x_F)} \geq A(F) \geq (1-O(\sqrt[4]{\varepsilon}))\cdot\frac{2\int_{\partial K}\kappa^{1/2}(x)dx}{n\kappa^{1/2}(x_F)}.$$

In summary, we have showed the following: $\exists\varepsilon_0 : \forall 0 < \varepsilon < \varepsilon_0 : \exists N(\varepsilon) > 0$ such that if $n > N(\varepsilon)$ then for all but $\sqrt[4]{\varepsilon}$ percent of the F faces of P_n^c we have that there exists a unique point $x_F \in \partial K$ such that $u(x_F)$ is an outer normal also to F, Q_{x_F} is positive definite, and F is $\sqrt[16]{\varepsilon}$-close to a square with respect to Q_{x_F} which is of area

$$\frac{2\int_{\partial K}\kappa^{1/2}(x)dx}{n\kappa^{1/2}(x_F)}.$$

This completes the proof of the theorem in the case of circumscribed polytopes.

In order to prove the theorem in the inscribed case, observe that every step of the argument can be repeated, with only two minor changes. One is that we have to apply the second part of Lemma 5 instead of the first, second that we use Lemma 3 instead of Lemma 2.

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