APPROXIMATING 3-DIMENSIONAL CONVEX BODIES BY POLYTOPES WITH A RESTRICTED NUMBER OF EDGES

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This paper is dedicated to Gábor Fejes Tóth on occasion of his sixtieth birthday.

ABSTRACT. We prove an asymptotic formula for the Hausdorff distance of a 3-dimensional convex body K with a C^2 boundary and its best approximating circumscribed polytope whose number of edges is restricted.

1. NOTATIONS AND CONVENTIONS

We shall work in d-dimensional Euclidean space $\mathbb{E}^d$, with origin o, and scalar product $\langle\cdot,\cdot\rangle$, and induced norm $|\cdot|$. We shall not distinguish between the Euclidean space and the underlying vector space, and we will use the words *point* and *vector* interchangeably, as we need them. Points of $\mathbb{E}^d$ are denoted by small-case letters of the roman alphabet, and sets by capitals. For reals we use either Greek letters or small-case letters. For a compact convex set K, we write aff K to denote its affine hull, and relint K for its relative interior. A compact convex set K with nonempty interior is called a convex body. If the dimension of K is two, then we call it a *convex disc*. For the sake of brevity, we shall use the term *unit disk* for the unit radius circular disc. B^d stands for the unit ball in $\mathbb{E}^d$ centred at the origin. Volume in $\mathbb{E}^d$ is denoted by $V(\cdot)$, and two-dimensional Hausdorff measure is denoted by $A(\cdot)$. Let A, B be subsets of $\mathbb{E}^d$, then the *convex hull* of A and B is denoted by $[A, B]$.

There are numerous ways to define metrics on the space of convex bodies $\mathcal{K}^d$, of which the Hausdorff metric is one of the most natural and applicable ones. For $K, L \in \mathcal{K}^d$ the *Hausdorff distance* is defined by

$$\delta_H(K,L) = \min\{\lambda \geq 0 \mid K \subset L + \lambda B^d, L \subset K + \lambda B^d\}.$$

Then δ_H is a metric on $\mathcal{K}^d$, called the *Hausdorff metric.* For further details on convex sets and related measures consult the monographs of R. Schneider [24] and P.M. Gruber [19].

Let K be a convex body in $\mathbb{E}^d$, and let ∂K denote its boundary. We always integrate on ∂K with respect to the $(d-1)$-dimensional Hausdorff measure. We say that K has a C^2 boundary if for any $x \in \partial K$, a neighbourhood of x in ∂K is the graph of a convex C^2 function f that is defined in the orthogonal projection of that neighbourhood into the tangent plane T_x at x. For $x \in \partial K$ we write Q_x to denote the second fundamental form at x which is, in fact, the quadratic form representing

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the second derivative of f at x. Because of the convexity of K, Q_x is positive semi-definite. Its eigenvalues are the principal curvatures, and its determinant $\kappa(x)$ is the *Gauss-Kronecker curvature* of ∂K at x. Clearly, $\kappa(x) \geq 0$ for all $x \in \partial K$. If, in addition, $\kappa(x) > 0$ for all $x \in \partial K$, then we say that the boundary of K is C^2_+.

Throughout the paper we shall use the customary notations for the magnitude of functions. Let f and g be functions of positive integers, then we write $f(n) = O(g(n))$ if there exists a constant c depending on the given convex body K such that $|f(n)| \leq c \cdot g(n)$ for all $n \geq 1$, and $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$. Furthermore, we write $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$.

2. History and main result

Polytopal approximation of convex bodies with smooth boundary has been investigated intensively for over thirty years. The starting point of most work on asymptotic approximation of convex bodies was the monograph of L. Fejes Tóth [8], and the first major result about higher dimensions is due to R. Schneider [22].

Let $K \subset \mathbb{E}^d$ be a convex body with a C^2 boundary, and $0 \leq k \leq d-1$ an integer. One of the most often studied questions is how well one can approximate K by a polytope with a restricted number of k-faces. The same problem is also studied for approximating polytopes that are inscribed in K or circumscribed about K. In this paper, we concentrate on approximation in the Hausdorff metric, however, other metrics, like the L_1 metric, the Banach-Mazur metric, the symmetric difference metric and Schneider's distance also play important roles in the theory of polytopal approximation.

Let P_n denote a polytope minimising $\delta_H(K, P_n)$ under the condition that P_n has n k-faces. The existence of such a polytope clearly follows from the compactness of K and the continuity of the Hausdorff metric. If $k = 0$ or $k = d-1$, that is, when the number of vertices or facets is given, the following asymptotic formula is known.

$$\delta_H(K, P_n) \sim c \cdot \left(\int_{\partial K} \kappa(x)^{\frac{1}{2}} \, dx \right)^{\frac{2}{d-1}} \cdot n^{\frac{-2}{d-1}}, \tag{1}$$

where the constant c depends only on k and d. This formula is the combined result of the works of R. Schneider, P.M. Gruber and K. Böröczky Jr. To see more details consult [15], [16] and [18]. However, no asymptotic formula is known for the case when $1 \leq k \leq d-2$. A partial result by K. Böröczky Jr. [3] states that if the boundary of K is C^2_+ and n is large, then

$$c_1 \cdot \left(\int_{\partial K} \kappa(x)^{\frac{1}{2}} \, dx \right)^{\frac{2}{d-1}} \cdot n^{\frac{-2}{d-1}} < \delta_H(K, P_n) < c_2 \cdot \left(\int_{\partial K} \kappa(x)^{\frac{1}{2}} \, dx \right)^{\frac{2}{d-1}} \cdot n^{\frac{-2}{d-1}}, \tag{2}$$

where $c_1, c_2 > 0$ depend only on k and d. Similar asymptotic formulae and estimates are known when P_n is inscribed in K or circumscribed about K; for details see [15], [16] and [18].

In this article, we shall investigate inequality (2) for $d = 3$. Let K be a convex body in $\mathbb{E}^3$ with a C^2 boundary, and let $\mathcal{P}^c_n(K)$ denote the set of convex polytopes circumscribed about K and having at most n edges. There exists a polytope P^c_n, not unique in general, such that

$$\delta_H(K, P^c_n) = \inf\{\delta_H(K, P) : P \in \mathcal{P}^c_n(K)\}.$$

Similarly, there exists a, not necessarily unique, polytope P_n^i inscribed in K having at most n edges and minimising the Hausdorff distance $\delta_H(K, P_n^i)$. Finally there exists a, not necessarily unique, polytope P_n in $\mathbb{E}^3$ having at most n edges and minimising the Hausdorff distance $\delta_H(K, P_n)$. We shall prove the following asymptotic formulae.

Theorem. *If $K \in \mathcal{K}^3$ is a convex body with a C^2 boundary, then we have*

$$\delta_H(K, P_n^c), \delta_H(K, P_n^i) \sim \frac{1}{2} \int_{\partial K} \kappa^{1/2}(x)dx \cdot \frac{1}{n} \quad \text{as } n \to \infty. \tag{3}$$

Note that (3) is an improvement on inequality (2), and it provides a similar result to (1) for $d = 3$ and $k = 1$.

We present the details of the proof for the circumscribed case and only sketch the necessary changes for the inscribed case at the end of the paper. The proof for P_n^c will consist of two parts. First, we construct a polytope with a prescribed number of edges circumscribed about K which approximates K well in the Hausdorff metric. Second, we prove a lower bound on $\delta_H(K, P_n^c)$. We will achieve this estimate by transferring the problem to the plane, where estimates based on the second moment will be investigated.

Note that in order to prove (3) for P_n^c, it is enough to see that there exist a $\tau > 0$ and an $\varepsilon_0 > 0$ absolute constants with the property that for all $\varepsilon_0 > \varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ such that for all $n > n_0$ we have

$$\frac{1 - \tau\varepsilon}{2n} \int_{\partial K} \kappa^{1/2}(x)dx \le \delta_H(K, P_n^c) \le \frac{1 + \tau\varepsilon}{2n} \int_{\partial K} \kappa^{1/2}(x)dx \, .$$

In the course of the proof of the upper bound, we will construct two polyhedral surfaces with a prescribed number of edges, one over a subset of ∂K where the curvature κ is small, and another one over the rest of ∂K where κ is strictly positive. We shall refer to the former part of ∂K as the *flat part* and the latter the *curved part*. We will see that the polyhedral surface over the flat part has very few edges compared to the one over the curved part. Finally, we shall establish that the two polyhedral surfaces can be joined to form a polytope circumscribed about K without creating a large number of extra edges. Furthermore, when investigating the lower bound, we can ignore the flat part of ∂K.

3. Dividing the boundary

If the curvature on ∂K is allowed to be zero then we will separate the "flat" and the "curved" part of ∂K. To approximate the flat part of ∂K, we use Lemma 1 and Corollary 1 below, which are both based on Lemma 1 proved in [2].

A Jordan measurable open subset of the boundary of a convex polytope P is called a polytopal hyper-surface. If we have a Jordan measurable, open subset Z of ∂K then we define $Y \subset \partial P$ as the set of points y such that there exists an exterior normal to P at y which is the exterior normal to K at some $z \in Z$. We say that the polytopal hyper-surface Y in ∂P approximates Z. For $\beta > 0$ let $\Sigma(\beta)$ denote the set of points on ∂K where the minimal principal curvature is less then β.

Lemma 1 ([2]). *Fix $\varepsilon > 0$. Then for any small β where $\Sigma(\beta)$ is Jordan measurable, $\Sigma(\beta)$ satisfies the following property: For large m, there exists a polytopal hypersurface $Y_m \subset K$ and with at most m vertices approximating $\Sigma(\beta)$ such that*

$$\delta_H(\Sigma(\beta), Y_m) \leq \frac{\varepsilon^2}{m}\ .$$

Applying Lemma 1 with $\varepsilon/2$ in place of ε, and dilating Y_m from a fixed point in $\mathrm{int}K$ by a suitable factor slightly larger than one we deduce the following statement.

Corollary 1. *Lemma 1 holds with the same conditions not only if $Y_m \subset K$ but also when $Y_m \cap \mathrm{int}K = \emptyset$.*

For a given $\varepsilon > 0$, we separate the "flat" part of ∂K, which depends on ε. Let us choose $\beta > 0$ such that $\Sigma(\beta)$ is small enough to satisfy the conditions of Lemma 1 and Corollary 1 and

$$\int_{\Sigma(\beta)} \kappa^{1/2}(x)dx < \frac{\varepsilon}{2} \int_{\partial K} \kappa^{1/2}(x)dx.$$

We define

$$X' = \partial K \backslash \mathrm{cl}\Sigma(\beta). \tag{4}$$

Now, there exist relatively open Jordan-measurable subsets X and X_0 of ∂K satisfying the following conditions.

i) cl $X \subset$ relint X' and cl $X' \subset$ relint X_0;
ii) There exists $\eta = \eta(\varepsilon) > 0$ such that all principal curvatures at $x \in$ cl X_0 are all least η;
iii) $\int_X \kappa^{1/2}(x)dx > (1-\varepsilon)\int_{\partial K} \kappa^{1/2}(x)dx$.

We write $u(z)$ to denote the exterior unit normal at $z \in \partial K$. Let C be a convex polygon tangent to K at $x \in \mathrm{relint}\, C$ such that the orthogonal projection of $\mathrm{int}\, K$ into $\mathrm{aff}\, C$ covers C. For any function $f : C \to \mathbb{R}$, we define its graph

$$\Gamma(f) = \{z - f(z)u(x) : z \in C\}.$$

Let $f_C : C \to \mathbb{R}$ denote the convex C^2 function with $\Gamma(f_C) \subset \partial K$. We shall use q_y to denote the quadratic form representing the second derivative of f_C at $y \in C$, hence $q_x = Q_x$ (q_y naturally depends on C, as well). In addition $p_C : \mathbb{E}^3 \to \mathrm{aff}\, C$ denote the orthogonal projection onto $\mathrm{aff}\, C$, and $\Pi_{\partial K} : C \to \partial K$ the nearest point map onto ∂K.

Lemma 2. *For each $\varepsilon > 0$, there exists a $\delta(K, \varepsilon) = \delta > 0$ such that if $C \subset x + \delta B^3$ is a convex polygon touching K at $x \in X' \cap \mathrm{relint}C$, then the following statements hold. We have $\Gamma(f_C) \subset X_0$, and*

$$\text{for all } y \in C,\ (1+\varepsilon^3)^{-1}Q_x \leq q_y \leq (1+\varepsilon^3)Q_x, \tag{5}$$

$$\text{for all } z \in \Gamma(f_C),\ \langle u(z), u(x)\rangle \geqq (1+\varepsilon)^{-1} \tag{6}$$

$$p_C(\Pi_{\partial K}(C)) \supseteq (1-\varepsilon)(C - x) + x. \tag{7}$$

Proof. The condition (6) and (7) holds for small enough $\delta > 0$ by the continuity of $u(z)$. Therefore we only consider (5).

For all $x \in \mathrm{cl}X'$ there exists a $\delta_x > 0$ such that (5) holds with $\varepsilon^3/10$ in place of ε^3. Let $B_x = x + (\delta_x/4)B^3$. Then

$$\bigcup_{x \in \text{ cl } X'} B_x \supseteq \text{cl } X'.$$

Since cl X' is compact, there exists a finite set $V \subset$ cl X' such that

$$\bigcup_{x \in V} B_x \supseteq \text{cl } X'.$$

Let $\delta > 0$ such that

$$\delta \leq \frac{\min_{x \in V} \delta_x}{4} \; .$$

and (6) holds with $\varepsilon^3/10$ instead of ε for this δ.

Now, let $x \in$ cl X' be an arbitrary point, let $C \subset x + \delta\, B^3$ be a convex polygon touching K at $x \in \mathrm{relint} C$. Then there exists an $\tilde{x} \in V$ such that $x \in B_{\tilde{x}}$, and a convex polygon $\widetilde{C} \subset \tilde{x} + \delta_{\tilde{x}} B^3$ touching K at $\tilde{x}$ with $p_{\widetilde{C}}(C \cup \Gamma(f_C)) \subset \widetilde{C}$.

For $y \in C$ and $z = y - f_C(y)u(x)$, we have $\langle u(x), u(\widetilde{x})\rangle \geqq (1+\varepsilon^3/10)^{-1}$. We write y_* to denote the orthogonal projection of z into $\widetilde{C}$, and write q and $\tilde{q}$ to denote the quadratic forms representing the second derivatives of f_C and $f_{\widetilde{C}}$. If v is a vector parallel to aff C and v' is its orthogonal projection into the linear two-dimensional subspace parallel to aff $\widetilde{C}$, then

$$(1+\varepsilon^3/10)^{-3}\tilde{q}_{y_*}(v') \leq q_y(v) \leq (1+\varepsilon^3/10)^3\tilde{q}_{y_*}(v').$$

Since $Q_x = q_x$, we obtained (5). □

4. The Momentum Lemmas

As a first step towards the proof of the Theorem, in this section we shall establish one of the major tools to be used subsequently, the Momentum Lemma. In fact, we will prove two slightly different versions of it in the following two statements.

Lemma 3. *Let $q(x)$ be a positive definite quadratic form on $\mathbb{E}^2$, and $\alpha \leqq 0$ a real number. If Π is a polygon with at most k sides, then*

$$\max_{x \in \Pi}(q(x) - \alpha) \geqq \frac{2}{k} \cdot A(\Pi)\sqrt{\det q}$$

with equality if $\alpha = 0$ and Π is a square with respect to q centred at o.

Proof. We may assume that $\alpha = 0$ and $q(x) = x^2$.

Step 1. Let T be a right triangle with an angle $\varphi < \pi/2$ at o. If c denotes the hypotenuse of T, then

$$\max_{x \in T} q(x) = c^2 = 2 \cdot \frac{c}{2} \cdot c \geqq 2 \cdot m_c \cdot c = 4 \cdot A(T).$$

Step 2. Suppose that the triangle $\hat{T}$ is obtuse, that is, it has an angle $\beta > \pi/2$, and an angle $\varphi < \pi/2$ at o. Let c denote the longest side of $\hat{T}$. Then we can complete $\hat{T}$ into a right triangle $\overline{T}$ which satisfies the conditions of Step 1. It follows that

$$\max_{x \in \bar{T}} q(x) = c^2 = \max_{x \in T} q(x) \geqq 4 \cdot A(T) \geqq 4 \cdot A(\bar{T}).$$

Step 3. Suppose that $o \in \Pi$. In this case, we may cut Π into triangles $T_1, \dots, T_m$, where $m \leqq 2k$, and each T_i is either like T or $\hat{T}$. Using the results of Steps 1 and 2, we obtain

$$\max_{x\in\Pi} q(x) \geqq \max_{i\in[m]} \max_{x\in T_i} q(x) \geqq 4 \cdot \max_{i\in[m]} A(T_i) \geqq 4 \cdot \frac{A(\Pi)}{2k}.$$

Step 4. Finally, let $o \notin \Pi$. If p denotes the point of Π that is nearest to the origin, then

$$\max_{x\in\Pi} x^2 \geqq \max_{x\in\Pi} (x-p)^2 \geqq \frac{2}{k} \cdot A(\Pi),$$

where the last inequality follows from Step 3.

Finally, if Π is a square centred at o and $\alpha = 0$ then we have equality in the statement. This completes the proof of Lemma 3. □

Lemma 4. *Let $q(x)$ be a positive definite quadratic form. If Π is a polygon with at most k sides, and $\alpha \geqq 0$ is a real number such that $q(x) \leqq \alpha$ for all $x \in \Pi$, then*

$$\max_{x\in\Pi}(\alpha - q(x)) \geqq \frac{2}{k} \cdot A(\Pi) \cdot \sqrt{\det q}$$

with equality if Π is a square with respect to q centred at o and $\alpha = q(x)$ for some vertex x of Π.

Proof. We may suppose that $q(x) = x^2$.

Step 1. Let T be a triangle, which has an angle $\beta \geqq \pi/2$, and an angle $\varphi < \pi/2$ at o. Let c denote the longest side of T. From $x^2 = q(x) \leqq \alpha$ it follows that $c^2 \leqq \alpha$. One can see that

$$A(T) \leqq \frac{c^2}{4} \leqq \frac{\alpha}{4} = \frac{\max_{x\in T}(\alpha - x^2)}{4}.$$

Step 2. Suppose that $o \in \Pi$. In this case, we may cut Π into triangles $T_1, \dots, T_m$, where $m \leqq 2k$ and T_i is of the same type as T in Step 1. Using Step 1, we obtain

$$\max_{x\in\Pi}(\alpha - q(x)) \geqq \max_{i\in[m]} \max_{x\in T_i}(\alpha - q(x)) \geqq 4 \cdot \max_{i\in[m]} A(T_i) \geqq 4 \cdot \frac{A(\Pi)}{2k}.$$

Step 3. Suppose that $k \geqq 4$, and $o \notin \Pi$. Let p be the nearest point of Π to the origin, and $d = |\overrightarrow{op}|$. Then the circle of radius $\sqrt{\alpha - d^2}$ centred at p contains Π. Furthermore, there exists a line through p which separates Π from the origin. We conclude that

$$\max_{x\in\Pi}(\alpha - q(x)) = \alpha - d^2 \geqq \frac{2}{k} \cdot \frac{\pi}{2}(\alpha - d^2) \geqq \frac{2}{k} \cdot A(\Pi).$$

Step 4. The last case to be checked is when Π is a triangle and $o \notin \Pi$. Using the same notations as in Step 3, we obtain

$$\max_{x\in\Pi}(\alpha - q(x)) = \alpha - d^2 = \frac{2}{3} \cdot \left(\frac{3}{2}(\alpha - d^2)\right) \geqq$$
$$\frac{2}{3}\sqrt{\alpha - d^2}\,(\sqrt{\alpha} - d) \geqq \frac{2}{3} A(\Pi).$$

Finally, if Π is a square centred at o and $\alpha = x^2$ for some vertex x of Π then we have equality in the statement. This completes the proof of Lemma 4. □

5. Construction and upper bound

In this section, we shall construct a polytope R_n^c with n edges circumscribed about K in such a way that it is close to K in the Hausdorff metric. As it was mentioned before, we are going to use two different constructions.

One of our frequent tools is the Taylor formula which we will use in a special form. Let f be a real C^2 function on a convex polygon C, let l_z be the linear form representing the derivative of f at z, and let q_z be the quadratic form representing the second derivative of f at z. Now if $a, z \in C$ then there exists a $t \in (0,1)$ such that

$$f(z) = f(a) + l_a(z-a) + \tfrac{1}{2} q_{a+t(y-a)}(y-a). \tag{8}$$

Now, we are ready to prove the upper bound.

Lemma 5. *There exist a positive ε_0 and c depending on K with the following property. If $0 < \varepsilon < \varepsilon_0$ and $n > N$, where N depends on ε and K, then there exists a polytope R_n^c with at most $(1+c\varepsilon)n$ edges containing K such that*

$$\frac{\delta_H(K, R_n^c)}{\frac{1}{2n}\int_{\partial K} \kappa^{1/2}(x)dx} \leqq 1 + c \cdot \varepsilon. \tag{9}$$

Proof. To simplify the notation we introduce

$$I = \int_{\partial K} \kappa^{1/2}(x)dx.$$

For any $x \in X'$ and $\varrho > 0$, let T_x denote the tangent hyperplane to ∂K at x, and let

$$\mathcal{E}(x, \varrho) = \{z \in T_x : \tfrac{1}{2}\, Q_x(z-x) \leq \varrho^2\}.$$

In particular, the ellipse $\mathcal{E}(x,\varrho)$ is a circular disc of radius ϱ with respect to $\frac{1}{2}\, Q_x$. According to Section 3 ii), both principal curvatures of Q_x are at least $\eta > 0$ depending on ε. Therefore, if $\varrho \leq \frac{1}{4}\,\delta\sqrt{\eta}$ then the Euclidean diameter of $\mathcal{E}(x,\varrho)$ is at most δ, where δ is the same as in Lemma 2.

Fix an $\varepsilon > 0$. Let $y \in X'$ be arbitrary. We are going to construct a polytopal surface near the point y that will be used in obtaining R_n^c. Consider the quadratic form $S_y = \frac{1-\varepsilon^3}{2}\, Q_y$. The graph of $S_y(z-y)$ (as a function of $z \in T_y$) above T_y is a paraboloid surface Ω, whose part above $T_y \cap (y + \delta B^3)$ does not intersect int$K$ by Lemma 2. Let us consider a side-to-side tiling of T_y by parallelograms that are squares with respect to Q_y and have area $2I/(n\kappa(y))^{1/2}$. We call this tiling a square grid. For a tile N in the square grid, we define a_N to be the centre of N. The size of N is chosen in a way such that (compare Lemma 3)

$$\max_{z\in N} \tfrac{1}{2}\, Q_y(z-a_N) = \frac{I}{2n} = \int_{\partial K} \kappa^{1/2}(x)dx \cdot \frac{1}{2n}. \tag{10}$$

We define $\nu = \frac{\sqrt{I}}{\sqrt{n}\cdot\varepsilon}$, hence

$$N - a_N \subset \mathcal{E}(y, \varepsilon\nu). \tag{11}$$

We assume that n is large enough to ensure that the Euclidean diameter of $\mathcal{E}(y, 12\nu)$ is at most δ.

We write $\widetilde{\Lambda}'$ to denote the family of squares in the grid that lie in $\mathcal{E}(y, 12\nu)$. Let $\omega = \omega_y$ and $f = f_y$ be the convex functions on $\cup\widetilde{\Lambda}'$ such that $\omega(z) = S_y(z-y)$, and $\Gamma(f) \subset \partial K$. It follows by the Taylor formula (8) that there exists a convex piecewise linear function $\varphi = \varphi_y$ on $\cup\widetilde{\Lambda}'$ such that φ is linear on each tile $N \in \widetilde{\Lambda}'$ with $\varphi(a_N) = \omega(a_N)$, and the graph of the linear function touches Ω. Let $\widetilde{\Lambda}_y = \Gamma(\varphi)$ be the corresponding polytopal surface, hence its faces project into the tiles in $\widetilde{\Lambda}'$. We are going to give an upper bound for the Hausdorff-distance between $\Gamma(f)$ and $\widetilde{\Lambda}_y$.

It follows from (10) and the Taylor formula (8) that

$$\delta_H(\widetilde{\Lambda}_y, \Gamma(\omega)) \leq \max_{N\in\widetilde{\Lambda}'} \max_{z\in N} (\omega(z) - \varphi(z)) = \max_{N\in\widetilde{\Lambda}'} \max_{z\in N} S_y(z - a_N) \leq \frac{\int_{\partial K} \kappa^{1/2}(x)dx}{2n}. \tag{12}$$

Next, we consider the distance between the paraboloid and the boundary of K. Combining Lemma 2 and the Taylor formula (8) yield

$$\begin{aligned}\delta_H(\Gamma(f), \Gamma(\omega)) &\leq \max_{z\in\mathcal{E}(x,12\nu)} (f(z) - \omega(z)) \leq \max_{z\in\mathcal{E}(x,12\nu)} \left(\tfrac{1+\varepsilon^3}{2}\, Q_y(z-y) - S_y(z-y)\right)\\ &= \max_{z\in\mathcal{E}(x,12\nu)} \varepsilon^3 Q_y(z-y) = 2\cdot 12^2 \nu^2 \varepsilon^3 = \frac{288 I\varepsilon}{n}. \end{aligned} \tag{13}$$

Therefore we conclude

$$\delta_H(\widetilde{\Lambda}_y, \Gamma(f_y)) \leq (1 + O(\varepsilon)) \cdot \frac{\int_{\partial K} \kappa^{1/2}(x)dx}{2n}. \tag{14}$$

Now we are going to construct the polytope R_n^c. Let $y_1, \ldots y_k$ be a maximal family of points of X' such that the sets $\Pi_{\partial K}(\mathrm{relint}\mathcal{E}(y_i, \nu))$ are pairwise disjoint. We define M to be the (possibly unbounded) polyhedral set determined by the tangent planes $T_{y_1}, \ldots, T_{y_k}$. For the face C_i' of M touching at y_i, we define $C_i = C_i' \cap \mathcal{E}(y_i, 4\nu)$. If n is large enough then the sets $\Pi_{\partial K}(C_1), \ldots, \Pi_{\partial K}(C_k)$ do not overlap and cover X'. For $i = 1, \ldots, k$, let Λ_i be the union of all faces F of $\widetilde{\Lambda}_{y_i}$ satisfying $\Pi_{\partial K} F \cap \Pi_{\partial K} C_i \neq \emptyset$, hence $p_{C_i}\Lambda_i \subset \mathcal{E}(x, 5\nu)$. Finally, to approximate the flat part, we choose $m = \varepsilon n$, and consider the Y_m approximating $\Sigma(\beta) = \partial K \setminus \mathrm{cl} X'$ provided by Corollary 1. Let Λ_0 be the union of all faces F with $\Pi_{\partial K} F \cap \Sigma(\beta) \neq \emptyset$. We define

$$R_n^c = [\Lambda_0, \Lambda_1, \ldots, \Lambda_k]. \tag{15}$$

The way how $\Lambda_0, \Lambda_1, \ldots, \Lambda_k$ were constructed, Corollary 1 and (14) yield directly that

$$\begin{aligned} K &\subset R_n^c,\\ \delta_H(K, R_n^c) &\leq (1 + O(\varepsilon)) \cdot \frac{\int_{\partial K} \kappa^{1/2}(x)dx}{2n}. \end{aligned} \tag{16}$$

Therefore all we are left to do is to estimate the number of edges of R_n^c.

Let G be the graph defined by the edges of R_n^c that are intersections of two faces of Λ_i for some $i \geq 1$, and let G' be the graph defined by the rest of the edges.

First, we estimate the number $e(G)$ of edges of G. Let $i = 1, \ldots, k$. We write m_i to denote the number of elements of Λ_i, and observe that

$$\mathcal{E}(y_i, \tfrac{1}{2}\,\nu) \subset C_i \subset \mathcal{E}(y_i, 4\nu). \tag{17}$$

We deduce from Lemma 2 and (11) that

$$p_{C_i}\Lambda_i \subset y_i + (1 + 4\varepsilon)(C_i - y_i). \tag{18}$$

Using the fact that $A(p_{C_i}F) = 2I/(n\kappa(y_i)^{1/2})$ for a face F of Λ_i and Lemma 2, we have that

$$m_i \le (1+O(\varepsilon)) \cdot \frac{n\kappa(y_i)^{1/2}A(C_i)}{2I} = (1+O(\varepsilon)) \cdot \frac{n}{2} \cdot \frac{\int_{\Pi_{\partial K}C_i} \kappa(x)^{1/2}dx}{\int_{\partial K} \kappa(x)^{1/2}dx}. \tag{19}$$

Since each face of Λ_i is a quadrilateral, it follows that

$$e(G) \le \sum_{i=1}^{k} 2m_i \le (1+O(\varepsilon)) \cdot n. \tag{20}$$

Now we turn to G'. We first count the number of vertices of G'. For $i = 1, \dots, k$, let μ_i be the number of common vertices v of Λ_i and G' such that there exists an edge $[v, w]$ of G', where w is a vertex of some Λ_j with $j \ge 1$ (possibly $j = i$). In addition, let μ_0 be the number of vertices v of G' such that either v is a vertex of Λ_0 or there exists an edge $[v, w]$ of G', where w is a vertex of Λ_0.

Let us consider an edge $[v, w]$ of G', where v is a vertex of Λ_i for some $i \ge 1$. For $v' = p_{C_i}v$ and $w' = p_{C_i}w$, it follows from (16) that

$$0 \le g(t) = \omega_{y_i}(tw' + (1-t)v') - \varphi_{y_i}(tw' + (1-t)v') \le I/n \quad \text{for } t \in [0, 1].$$

Since $g(t) = \frac{1-\varepsilon^3}{2} Q_{y_i}(t(w'-v')) + at + b$ for some $a, b \in \mathbb{R}$, the inequalities $g(0) \le I/n$ and $g(1) - 2g(1/2) \le I/n$ yield

$$Q_{y_i}(w' - v') \le \tfrac{2}{1-\varepsilon^3} \cdot \tfrac{8I}{n} \le \tfrac{32I}{n}. \tag{21}$$

First, we assume that w is a vertex of some Λ_j with $j \ge 1$, and v is a vertex of a face F of Λ_i. It follows from (17) and by applying (18) to C_j with $j \ne i$ that

$$p_{C_i}F \cap (y_i + (1-\gamma\varepsilon)(C_i - y_i)) = \emptyset$$

for some absolute constant $\gamma > 0$. Using the fact that $A(p_{C_i}F) = 2I/(n\kappa(y_i)^{1/2})$ and (18) we may conclude that

$$\mu_i \le O(\varepsilon) \cdot \frac{n\kappa(y_i)^{1/2}A(C_i)}{2I} = O(\varepsilon) \cdot n \cdot \int_{\Pi_{\partial K}C_i} \kappa(x)^{1/2}dx. \tag{22}$$

Now we are going to estimate μ_0. If $[v, w]$ is an edge of G' such that v is a vertex of Λ_i for some $i \ge 1$ and w is a vertex of Λ_0, then the definition of Λ_0 and (21) yield $\Pi_{\partial K}C_i \cap X = \emptyset$. Since Λ_0 has at most εn vertices and Λ_i has at most $4m_i$ vertices for $i \ge 1$, we deduce using (19) and the (iii) in Section 3 that

$$\mu_0 \le \varepsilon n + \sum_{\Pi_{\partial K}C_i \cap X = \emptyset} 4m_i \le \varepsilon n + \frac{n \cdot \int_{\partial K \setminus X} \kappa(x)^{1/2}dx}{\int_{\partial K} \kappa(x)^{1/2}dx} \le 2\varepsilon n.$$

Combining the above estimate with (22), $i = 1, \dots, k$, shows that G' has at most $O(\varepsilon)n$ vertices. G' is a planar graph, therefore the number $e(G')$ of its edges is at most three times the number of its vertices (see P. Brass, W. Moser, J. Pach [6]). In particular, $e(G') = O(\varepsilon)n$. We deduce by (20) that R_n^c has at most $(1 + O(\varepsilon))n$ edges, which concludes the proof of Lemma 5. □

Corollary 2. *If K is a convex body in $\mathbb{R}^3$ with a C^2 boundary and P_n^c is a best approximating circumscribed polytope with respect to the Hausdorff metric with at most n edges, then*

$$\limsup_{n\to\infty} \frac{\delta_H(K, P_n^c)}{\frac{1}{2n} \cdot \int_{\partial K} \kappa^{1/2}(x)dx} \leqq 1.$$

6. Lower bound

Let K be the convex body with C^2 boundary of the main Theorem. In this section we shall prove that

$$\liminf_{n\to\infty}[n\cdot\delta_H(K,P_n^c)] \geqq \frac{1}{2}\int_{\partial K}\kappa^{1/2}(x)dx \tag{23}$$

It is enough to see that there exists an $\varepsilon_0>0$ (depending only on K) such that for every $\varepsilon_0>\varepsilon>0$ there is a positive integer N_ε with the property that if $n>N_\varepsilon$, then

$$\delta_H(K,P_n) \geqq \frac{(1-\tau\varepsilon)}{2}\int_{\partial K}(\kappa(x))^{1/2}dx\cdot\frac{1}{n}, \tag{24}$$

where τ is a constant which depends only on K.

6.1. **Transfer Lemma.** We are going to prove a statement that transfers the problem into the plane where we need only investigate certain integral expressions based on the second moment.

First, we establish the notations and conventions we will use in stating and proving Lemma 6. We use δ and X' from Section 3. For $x\in X'$, let $C\subset x+\delta B^3$ be a convex polygon such that C touches K at x such that x lies in the relative interior of C. Furthermore, let C' be a convex polygon such that $C'\subset \operatorname{relint} C$ and $C\subset p_C(K)$. We write f to denote the convex function on C such that $\Gamma(f)\subset\partial K$. We shall use l_y for the linear from representing the first derivative of f at $y\in C$, and, as usual, q_y to denote the quadratic form representing the second derivative of f at $y\in C$. Note that $Q_x=q_x$.

Next, let P be a polytope with $C\subset p_C(P)$, and let $F_1,\dots,F_k$ to denote the faces of P whose exterior unit normal encloses an acute angle with $u(x)$ and satisfy $p_C(F_i)\cap C'\neq\emptyset$, $i=1,\dots,k$. Furthermore, we assume that

$$p_C(F_i)\subset C,\ i=1,\dots,k,$$

and for any F_i, there exists an $a_i\in C$ such that the exterior unit normal to F_i coincides with the exterior unit normal to the graph of f at $a_i-f(a_i)u(x)$. In particular, aff F_i is the graph of the function

$$\varphi_i(y)=f(a_i)+l_{a_i}(y-a_i)+\alpha_i$$

for some $\alpha_i\in\mathbb{R}$. Finally, we define $\Pi_i=p_C(F_i)$.

Lemma 6 (Transfer lemma)**.** *Let $\varepsilon\in(0,1/4)$. Using the notation as above, we assume that Q_x is positive definite and for all $z\in C$, we have $(1+\varepsilon)^{-1}Q_x\leqq q_z\leqq(1+\varepsilon)Q_x$ and $\langle u(x),u(w)\rangle\geq(1+\varepsilon)^{-1}$ for $w=z-f(z)u(x)$.*

(i) *If $K\subset P$ then each $\alpha_i\leq 0$, and*

$$\delta_H(P,K)\geqq(1-2\varepsilon)\max_{i=1,\dots,k}\max_{y\in\Pi_i}\left(\tfrac{1}{2}\,Q_x(y-a_i)-\alpha_i\right).$$

(ii) *If $P\subset K$ then each $\alpha_i>0$. Letting $\alpha_i'=(1+\varepsilon)\alpha_i$ for $i=1,\dots,k$, we have $\frac{1}{2}\,Q_x(y-a_i)\leq\alpha_i'$ for $y\in\Pi_i$, $i=1,\dots,k$, and*

$$\delta_H(P,K)\geqq(1-4\varepsilon)\max_{i=1,\dots,k}\max_{y\in\Pi_i}\left(\alpha_i'-\tfrac{1}{2}\,Q_x(y-a_i)\right).$$

Proof. For $i = 1, \dots, k$, we define $\Pi_i' = [\Pi_i, \{a_i\}]$.

First, let $K \subset P$, then $f(y) \geq \varphi_i(y)$ for $y \in C$ and $i = 1, \dots, k$. We claim that

$$(1+\varepsilon)\delta_H(P,K) \geqq \max_{i=1,\dots,k} \max_{y\in\Pi_i'}(f(y) - \varphi_i(y)). \tag{25}$$

Since $f(y) - \varphi_i(y)$ is convex, we may assume that y is a vertex of Π_i', and as $f(y) - \varphi_i(y)$ attains its minimum at a_i, we may assume that $y \in \Pi_i$. Let $w = y - \varphi_i(y)u(x) \in F_i$, and let T be the tangent plane at $v = y - f(y)u(x) \in \partial K$. Since T separates K and w and the exterior unit normal $u(v)$ satisfies $\langle u(x), u(v)\rangle \geq (1+\varepsilon)^{-1}$, we have

$$\delta_H(P,K) \geq d(w,T) \geq (1+\varepsilon)^{-1}(f(y) - \varphi_i(y)),$$

where $d(w,T)$ is the distance of w and T. The above inequality implies (25).

For any $y \in \Pi_i$ there exists $z \in \Pi_i' \subset C$ such that $f(y) - \varphi_i(y) = \frac{1}{2}\, q_z(y-a_i) - \alpha_i$ according to the Taylor formula (8), where $q_z(y-a_i) - 2\alpha_i \leq 2(1+\varepsilon)\delta_H(P,K)$ by (25). Therefore

$$\begin{aligned}(1+\varepsilon)\delta_H(P,K) &\geqq \tfrac{1}{2}\, q_z(y-a_i) - \alpha_i\\ &\geq \tfrac{1}{2}\, Q_x(y-a_i) - \alpha_i - \tfrac{1}{2}|q_z(y-a_i) - Q_x(y-a_i)|\\ &\geq \tfrac{1}{2}\, Q_x(y-a_i) - \alpha_i - \varepsilon\, \delta_H(P,K).\end{aligned}$$

Thus, we have verified Lemma 6 i).

Next, let $P \subset K$, hence $\varphi_i(y) \geq f(y)$ for $y \in C$ and $i = 1, \dots, k$. We claim that

$$(1+\varepsilon)\delta_H(P,K) \geqq \max_{i=1,\dots,k} \alpha_i \geq \max_{i=1,\dots,k} \max_{y\in\Pi_i'}(\varphi_i(y) - f(y)). \tag{26}$$

The second inequality is a consequence of the facts that $\varphi_i(y) - f(y)$ is concave and $\varphi_i(a_i) - f(a_i) = \alpha_i$. To prove the first inequality in (26), set $w = a_i - \varphi_i(a_i)u(x) \in \operatorname{aff} F_i$, and $v = a_i - f(a_i)u(x) \in \partial K$. As $\operatorname{aff} F_i$ separates P and v and its exterior unit normal $u(a_i)$ satisfies $\langle u(x), u(a_i)\rangle \geq (1+\varepsilon)^{-1}$, we have

$$\delta_H(P,K) \geq d(v, \operatorname{aff} F_i) \geq (1+\varepsilon)^{-1}(\varphi_i(a_i) - f(a_i)) = (1+\varepsilon)^{-1}\alpha_i,$$

which implies (26).

For any $y \in \Pi_i$ there exists $z \in \Pi_i' \subset C$ such that $\varphi_i(y) - f(y) = \alpha_i - \frac{1}{2}\, q_z(y-a_i)$ according to the Taylor formula (8), where $\alpha_i, q_z(y-a_i) \leq 2\delta_H(P,K)$ by (25). We deduce

$$\begin{aligned}(1+\varepsilon)\delta_H(P,K) &\geqq \alpha_i - \tfrac{1}{2}\, q_z(y-a_i)\\ &\geq \alpha_i' - \tfrac{1}{2}\, Q_x(y-a_i) - \tfrac{1}{2}|q_z(y-a_i) - Q_x(y-a_i)| - (\alpha_i' - \alpha_i)\\ &\geq \alpha_i' - \tfrac{1}{2}\, Q_x(y-a_i) - 3\varepsilon\, \delta_H(P,K).\end{aligned}$$

Therefore the proof of Lemma 6 is now complete. □

6.2. **Proof of the lower bound.** We are going to construct a "large" auxiliary polytope $M = M(\varepsilon)$ circumscribed about K. We require that M has the following property. If C is a face of M and $\Pi_{\partial K} C \subset X_0$ then $\operatorname{diam} C < \delta$, where δ is defined in Lemma 2. Let $\widehat{\mathcal{C}}$ be the family of all faces C of M such that $\Pi_{\partial K} C \cap X \neq \emptyset$. For all $C \in \widehat{\mathcal{C}}$, there is a unique $x_C \in X_0$ such that $u(x_C)$ is normal to C. Define

$$\mathcal{C} = \{x_C + (1-2\varepsilon)(C - x_C) \mid C \in \widehat{\mathcal{C}},\ \Pi_{\partial K}(x_C + (1-2\varepsilon)(C - x_C)) \cap X \neq \emptyset\}.$$

Properties (ii) and (iii) of X (see Section 3) and Lemma 2 yield

Lemma 7.

$$\sum_{C\in\mathcal{C}} \kappa^{1/2}(x_C)\cdot A(C) \geq (1-O(\varepsilon))\int_{\partial K}\kappa^{1/2}(x)dx$$

Corollary 2 and the property (ii) of X_0 (see Section 3) imply that there exists an $\omega=\omega(K,\varepsilon)>0$ such that if F is a face of P_n^c and $\exists C\in\widehat{\mathcal{C}}$ with $\Pi_{\partial K}(C)\cap\Pi_{\partial K}(F)\neq\emptyset$ then diam $F\leqq \omega/\sqrt{n}$. Now, if $n>N_\varepsilon$ then for every F there is at most one $C\in\mathcal{C}$ such that $\Pi_{\partial K}(F)\cap\Pi_{\partial K}(C)\neq\emptyset$. (Note that N_ε is independent of n.) Let us denote by $\mathcal{C}_C$ the set of those faces of P_n^c which are "above" C; namely, their projection into aff$C$ intersects C, and whose exterior unit normal encloses an acute angle with $u(x_C)$. For any $F\in\mathcal{C}_C$, let $\Pi_F=p_C(F)$, let $x_F\in\partial K$ satisfy that $u(x_F)$ is the exterior unit normal to F. Let k_F denote the number of sides of F. Now if n is large then $a_F=p_C(x_F)$ lies in C, and let α_F be defined by $x_F-\alpha_F u(x_C)\in$ aff$F$, hence $\alpha_F\leq 0$.

We deduce by Lemmas 6 and 3 (the Transfer and the Momentum lemmas), and by $\sqrt{\det\frac{1}{2}\,Q_{x_C}}=\frac{1}{2}\,\kappa^{1/2}(x_C)$ the estimates

$$\begin{aligned}\delta_H(P_n^c,K) &\geqq (1-O(\varepsilon))\max_{C\in\widehat{\mathcal{C}}}\max_{F\in\mathcal{C}_C}\max_{y\in\Pi_F}(\tfrac{1}{2}\,Q_{x_C}(y-a_F)-\alpha_F)\\ &\geqq (1-O(\varepsilon))\cdot\max_{C\in\widehat{\mathcal{C}}}\max_{F\in\mathcal{C}_C}\frac{\kappa^{1/2}(x_C)\cdot A(\Pi_F)}{k_F}. \qquad (27)\end{aligned}$$

To estimate the maximum on the right-hand side of (27) we need the following simple inequality.

Lemma 8. *Let $a_i,b_i\in\mathbb{R}_+$ for $i=1,2,\dots,m$. If there exists a λ such that $a_i/b_i\leqq\lambda$ holds for all $i=1,2,\dots,m$ then $(\sum_{i=1}^m a_i)/(\sum_{i=1}^m b_i)\leqq\lambda$.*

Proof. If we sum the inequalities $a_i\leqq\lambda b_i$ we get that $\sum a_i\leqq\sum\lambda b_i$. Dividing both side with $\sum b_i$ we obtain the statement of the lemma. □

Since the sum of the number of sides of the faces of P_n^c is at most $2n$, applying Lemmas 7 and 8 to (27), we obtain that

$$\begin{aligned}\delta_H(P_n^c,K) &\geqq (1-O(\varepsilon))\cdot\frac{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C}(\kappa^{1/2}(x_C)\cdot A(\Pi_F))}{\sum_{C\in\widehat{\mathcal{C}}}\sum_{F\in\mathcal{C}_C}k_F}\\ &\geqq (1-O(\varepsilon))\cdot\frac{\sum_{C\in\widehat{\mathcal{C}}}(\kappa^{1/2}(x_C)\cdot A(C))}{2n}\\ &\geqq \frac{1-O(\varepsilon)}{2}\int_{\partial K}\kappa^{1/2}(x)dx\cdot\frac{1}{n}.\end{aligned}$$

Since ε was arbitrary, we have finished the proof of (3), and thus the proof of the Theorem in the case of P_n^c. □

7. The inscribed case

In this section, we are going to prove formula (3) of the Theorem for P_n^i, that is for polytopes inscribed in K having at most n edges. The proof of this statement consists essentially of the same arguments as the proof for the circumscribed case, only minor modifications are necessary to make everything work. For the sake of brevity, we do not want to duplicate complete arguments, and thus we shall only point out the differences between the proofs.

To establish the lower bound, we are going to repeat the same argument as for circumscribed polytopes. There will only be one minor detail that differs. In the construction, we will face some complications with convexity, but these complications can be resolved with an extra step.

Now, we are going to give a sketch of the proof and emphasise only the important steps.

7.1. **Lower bound.** We use the same circumscribed auxiliary polytope M, and we would like to use Lemmas 6 and 4 (the Transfer and the Momentum lemmas). Using the same notations as in the circumscribed case, we proceed exactly as there until the definitions of α_F, etc. for a face $F \in \mathcal{C}_C$ of P_n^i. Now, $\alpha_F > 0$, and (in accordance with the Transfer Lemma) we define $\alpha'_F = (1+\varepsilon)\alpha_F$. Instead of (27), we deduce from Lemmas 6 and 4 the estimates

$$\begin{aligned} \delta_H(P_n^c, K) &\geqq (1 - O(\varepsilon)) \max_{C \in \widehat{\mathcal{C}}} \max_{F \in \mathcal{C}_C} \max_{y \in \Pi_F} (\alpha'_F - \tfrac{1}{2}\, Q_{x_C}(y - a_F)) \\ &\geqq (1 - O(\varepsilon)) \cdot \max_{C \in \widehat{\mathcal{C}}} \max_{F \in \mathcal{C}_C} \frac{\kappa^{1/2}(x_C) \cdot A(\Pi_F)}{k_F}. \end{aligned} \tag{28}$$

¿From this point on, the very same calculation works as in the circumscribed case.

7.2. **Upper bound.** We shall use the same notations as in Section 5. For $y \in X'$, in this case we define S_y by $S_y = \frac{1+\varepsilon^3}{2}\, Q_y$. Now, the graph of $S_y(z-y)$ (as a function of $z \in T_y$) above T_y is a paraboloid surface Ω whose part above $T_y \cap (y + \delta B^3)$ lies in K by Lemma 2. We define the square grid with respect to Q_y as in the circumscribed case, and define $\alpha = \frac{(1+\varepsilon^3)I}{2n}$. In particular (compare Lemma 4),

$$\max_{z \in N} S_y(z - a_N) = \alpha, \tag{29}$$

$$\max_{z \in N} (\alpha - S_y(z - a_N)) = \alpha = \frac{1+\varepsilon^3}{2n} \cdot \int_{\partial K} \kappa^{1/2}(x) dx. \tag{30}$$

Again, we write $\widetilde{\Lambda}'$ to denote the family of the squares in the grid that lie in $\mathcal{E}(x, 12\nu)$. Let ω and f be the convex functions on $\cup\widetilde{\Lambda}'$ such that $\omega(z) = S_y(z - y)$, and $\Gamma(f) \subset \partial K$. It follows from the Taylor formula (8) that there exists a convex piecewise linear function φ on $\cup\widetilde{\Lambda}'$ such that φ is linear on each tile $N \in \widetilde{\Lambda}'$ with $\varphi(a_N) = \omega(a_N) + \alpha$, and the graph $\widetilde{\Lambda}_y = \Gamma(\varphi)$ is inscribed into $\Gamma(\omega)$; namely, the vertices of $\widetilde{\Lambda}_y$ lie on $\Gamma(\omega)$. Similarly as in the circumscribed case, we deduce (14).

The rest of the proof works similarly to the argument in the circumscribed case with the small exception that we must use Lemma 1 instead of Corollary 1.

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