BEST AND RANDOM APPROXIMATIONS WITH GENERALIZED DISC–POLYGONS

KINGA NAGY AND VIKTOR VÍGH

ABSTRACT. In this paper, we consider the asymptotic behaviour of the distance between a convex disc K with sufficiently smooth boundary, and its approximating n-gons, as the number of vertices tends to infinity. We consider two constructions: the best approximating inscribed n-gon of K is the one with maximal area; and a random inscribed n-gon of K is the convex hull of n i.i.d. random points chosen from the boundary of K. The asymptotic behaviour of the area deviation of K and the n-gon depend in both cases on the same, geometric limit. The best and random approximating n-gons can be similarly defined in the circumscribed case.

We generalize the existing results on linear and spindle convexity to the so-called L-convexity. In the case of inscribed L-polygons, we prove similar asymptotic formulae by generalizing the geometric limits. Then we introduce an L-convex duality, consider its properties, and use them to prove the formulae for the circumscribed cases.

1. Introduction

In his famous book [5], László Fejes Tóth in connection with the affine arc-length provided some asymptotic formulae for best approximating polygons. In 1975, McClure and Vitale [12] gave a rigorous proof of these results, and developed a general analytical framework for investigating the asymptotic behaviour of best approximations of convex sets with a C^2_+ boundary. These formulae dealt with the best approximating inscribed and circumscribed polygons of a convex disc with at most n vertices with respect to perimeter deviation, area of the symmetric difference, and the Hausdorff metric. For instance, let K be a convex disc (that is convex, compact with non-empty interior) with a smooth enough boundary, and denote by K_n^A the n-gon contained in K that has maximal area. In other words the area deviation $\delta_A(K, K_n^A) = A(K) - A(K_n^A)$ is minimal. (Note that K_n^A is not unique in general.) Among others they proved that $\lim n^2 \cdot \delta_A(K, K_n^A) = \lambda^3(K)/12$, where $\lambda(K)$ denotes the affine perimeter of K. In 1999 Ludwig [13] proved the formulae regarding area deviation for general convex discs as well. These results have far reaching generalizations, and the notion of best approximating polytopes has been widely studied in the last 50 years, for a survey see for example [3].

Another notion of approximations arises by studying random polygons: take n independent, identically distributed random points from the boundary of the convex disc K. Consider their convex hull to obtain an inscribed polygon, and the intersection of the supporting half-planes at those points to obtain a circumscribed polygon. Schneider in [20] proved asymptotic formulae for the perimeter and area

Date: September 29, 2022.
2010 *Mathematics Subject Classification.* Primary 52A22, Secondary 60D05.
Key words and phrases. Disc-polygon, random approximation.

deviation of K and its approximating polygon, as well as for their Hausdorff distance. Later Reitzner generalized these results into higher dimensions [16]. The topic of random polytopes plays a central role in stochastic geometry, and has a really extensive literature, for a detailed historical review we refer to the recent survey by Schneider in [19] and also [6].

Both the above mentioned planar best and random approximating results depend on the same geometric limits. This relies on partitioning K and its approximating polygon at the vertices in the inscribed case, and at the points of tangency in the circumscribed case. The proofs of the generalisations take on a similar form. We refer to [10] for a comparison of best and random approximations.

The results of McClure and Vitale were also generalised by Fodor and Vígh in [9] from linear convexity to the so-called *spindle convexity* (see [2]). In this paper, our aim is to further extend these results, as well as the results by Schneider, to *L-convexity*, a notion introduced by Polovinkin [15], and independently by Lángi, Naszódi and Talata in [11]; see also [14] and [1].

Let L be a convex disc. To avoid some technical details, in what follows we also assume that K is a convex disc, however we note here that some definitions can be formulated in a slightly more general way. We say that K is *L-convex*, if it is equal to the intersection of all translates of L containing K (see [11, Definition 1.1]). For any set $X \subseteq \mathbb{R}^2$ contained in some translate of L, the *L-convex hull* of X, denoted by $[X]_L$, is the intersection of all translates of L containing X. We say that K is *L-spindle convex* ([11, Definition 1.2]), if it is contained in a translate of L, and contains the L-convex hull of any pair of its points, i.e. for every $x, y \in K$, $[x, y]_L \subseteq K$ holds as well. Clearly, if K is L-convex, then it is also L-spindle convex. In the planar case, the converse is also true, thus the two notions are equivalent. We note that in higher ($d \geq 3$) dimensions, the reverse implication fails.

2. Notation and results

We say that P is an *L-polygon* if it is the intersection of a finite number of L-translates, or equivalently, the L-convex hull of a finite number of points. The notion of vertex and edge are self-explanatory. Hence the construction of an inscribed L-polygon of K is evident. For the construction of a circumscribed L-polygon, we introduce the L-convex equivalent a supporting half-plane. By [11, Theorem 4], for any L-convex disc K and $x \in \partial K$, there exists a translate $L+p$ such that $K \subseteq L+p$, $x \in \partial(L+p)$ and K and $L+p$ share a support line at x. If the boundary of both K and L is differentiable, then the L-translate is unique at every boundary point of K. Then we say that $L+p$ is the *supporting* L of K at x, or $L+p$ *supports* K at x.

We now turn to defining the more general concepts of differential and convex geometry used in the paper. Let K be a convex disc with C^2 boundary, that is, ∂K is twice continuously differentiable. For any $u \in \mathcal{S}^1$, $x_K(u)$ denotes the unique boundary point of K with outer unit normal u, and $\kappa_K(u)$ and $r_K(u)$ denote the curvature and radius of curvature at $x_K(u)$, respectively. We say that ∂K is of class C^2_+, if it's C^2 and $\kappa_K(u) > 0$ for every $u \in \mathcal{S}^1$.

We define the *support function* of K as $h_K(u) = \sup_{x \in K}\langle x, u\rangle$ for every $u \in \mathcal{S}^1$. For the basic properties of the support function, we refer to [18, Section 1.7.1]. We note that h_K is twice differentiable, and $h_K(u) + h''_K(u) = r_K(u)$, see [17, Equation (1.5)].

For the sake of compactness, we use the notation $u(\theta) = (\cos\theta,\ \sin\theta)$ for the unit vector with angle θ. With a slight abuse of the notation we also write $x_K(\theta) = x_K(u(\theta))$, $\kappa_K(\theta) = \kappa_K(u(\theta))$, $r_K(\theta) = r_K(u(\theta))$ and $h_K(\theta) = h_K(u(\theta))$. Clearly, κ_K, r_K and h_K are 2π periodic in θ.

The *Hausdorff distance* of the convex sets K_1 and K_2 is defined as

$$\delta_H(K_1, K_2) = \max\left\{\sup_{x\in K_1}\inf_{y\in K_2}|x-y|, \sup_{y\in K_2}\inf_{x\in K_1}|x-y|\right\},$$

or equivalently,

$$\delta_H(K_1, K_2) = \inf\{\lambda \geq 0 \mid K_1 \subseteq K_2 + \lambda\mathcal{B}^2,\ K_2 \subseteq K_1 + \lambda\mathcal{B}^2\},$$

where

$$K + \lambda\mathcal{B}^2 = \{x + \lambda b \mid x \in K,\ b \in \mathcal{B}^2\}.$$

Then $\delta_H(\cdot,\cdot)$ is a metric on the set of convex discs, called the *Hausdorff metric*. For its further properties, see [18, Section 1.8]. It is worth mentioning that the Hausdorff distance of two sets can be expressed via their support functions:

$$\delta_H(K_1, K_2) = \max_{\theta\in[0,2\pi]}|h_{K_1}(\theta) - h_{K_2}(\theta)| \tag{1}$$

Furthermore we introduce the *area deviation* (also known as symmetric volume difference) and *perimeter deviation* of K_1 and K_2 as follows:

$$\delta_A(K_1, K_2) = A(K_1 \cup K_2) - A(K_1 \cap K_2) \quad \text{and}$$
$$\delta_P(K_1, K_2) = \mathrm{Per}(K_1 \cup K_2) - \mathrm{Per}(K_1 \cap K_2).$$

We note that δ_A is a metric on the space of convex discs, while δ_P is not, as the latter does not satisfy the triangle inequality.

Throughout the paper, both K and L denote convex discs with C^2_+ boundaries.

Let K_n^P, K_n^A and K_n^H ($K_{(n)}^P$, $K_{(n)}^A$ and $K_{(n)}^H$) denote the best approximating inscribed (circumscribed) L-polygon of K with at most n vertices, with respect to perimeter deviation, area deviation, and the Hausdorff metric, respectively. More precisely, let us denote by $\mathcal{P}_n^i$ ($\mathcal{P}_n^c$) the set of L-polygons contained in K (that contain K) that have at most n vertices. Now, there exist (not necessarily unique in general) L-polygons K_n^P, K_n^A, K_n^H, $K_{(n)}^P$, $K_{(n)}^A$ and $K_{(n)}^H$ such that

$$\delta_P(K, K_n^P) = \inf_{P\in P_n^i}\delta_P(K, P), \qquad \delta_P(K, K_{(n)}^P) = \inf_{P\in P_n^c}\delta_P(K, P),$$
$$\delta_A(K, K_n^A) = \inf_{P\in P_n^i}\delta_A(K, P), \qquad \delta_A(K, K_{(n)}^A) = \inf_{P\in P_n^c}\delta_A(K, P),$$
$$\delta_H(K, K_n^H) = \inf_{P\in P_n^i}\delta_H(K, P), \qquad \delta_H(K, K_{(n)}^H) = \inf_{P\in P_n^c}\delta_H(K, P).$$

Our main results are described in the following four theorems.

Theorem 1. *With the notation introduced above, the following hold.*

$$\lim_{n\to\infty} n^2 \cdot \delta_P\left(K, K_n^P\right) = \frac{1}{24}\left(\int_0^{2\pi}\left[\frac{r_K(\theta)}{r_L^2(\theta)}\cdot\left(r_L^2(\theta) - r_K^2(\theta)\right)\right]^{1/3}\mathrm{d}\theta\right)^3$$

$$\lim_{n\to\infty} n^2 \cdot \delta_A\left(K, K_n^A\right) = \frac{1}{12}\left(\int_0^{2\pi}\left[\frac{r_K^2(\theta)}{r_L(\theta)}\cdot\left(r_L(\theta) - r_K(\theta)\right)\right]^{1/3}\mathrm{d}\theta\right)^3$$

$$\lim_{n\to\infty} n^2 \cdot \delta_H\left(K, K_n^H\right) = \frac{1}{8}\left(\int_0^{2\pi}\left[\frac{r_K(\theta)}{r_L(\theta)}\cdot\big(r_L(\theta)-r_K(\theta)\big)\right]^{1/2}\mathrm{d}\theta\right)^2.$$

Theorem 2. *Let L be centrally symmetric. The following hold.*

$$\lim_{n\to\infty} n^2 \cdot \delta_P\left(K, K_{(n)}^P\right) = \frac{1}{24}\left(\int_0^{2\pi}\left[\frac{r_K^3(\theta)-3r_K^2(\theta)r_L(\theta)+2r_K(\theta)r_L^2(\theta)}{r_L^2(\theta)}\right]^{1/3}\mathrm{d}\theta\right)^3$$

$$\lim_{n\to\infty} n^2 \cdot \delta_A\left(K, K_{(n)}^A\right) = \frac{1}{24}\left(\int_0^{2\pi}\left[\frac{r_K^2(\theta)}{r_L(\theta)}\cdot\big(r_L(\theta)-r_K(\theta)\big)\right]^{1/3}\mathrm{d}\theta\right)^3$$

$$\lim_{n\to\infty} n^2 \cdot \delta_H\left(K, K_{(n)}^H\right) = \frac{1}{8}\left(\int_0^{2\pi}\left[\frac{r_K(\theta)}{r_L(\theta)}\cdot\big(r_L(\theta)-r_K(\theta)\big)\right]^{1/2}\mathrm{d}\theta\right)^2.$$

Let μ be a positive continuous density function on the boundary of K, and let $X_n = [x_1, \dots, x_n]$ be a random, independent sample of points from the boundary, distributed according to μ. We denote the convex hull $[X_n]_L$ of the sample by R_n, which is an inscribed L-polygon of K. Furthermore, $R_{(n)}$ denotes the circumscribed polygon of K obtained from X_n by taking the intersection of the support L of K at every point x_i $(i = 1, \dots, n)$. Lastly, we define the function m by $m(\theta) = \mu(x_K(\theta))/\kappa_K(\theta)$.

Theorem 3. *With the notation above, the following limits hold with probability* 1.

$$\lim_{n\to\infty} n^2 \cdot \delta_P\left(K, R_n\right) = \frac{1}{4}\int_0^{2\pi}\frac{r_K(\theta)}{r_L^2(\theta)}\cdot\big(r_L^2(\theta)-r_K^2(\theta)\big)\cdot m^{-2}(\theta)\mathrm{d}\theta$$

$$\lim_{n\to\infty} n^2 \cdot \delta_A\left(K, R_n\right) = \frac{1}{2}\int_0^{2\pi}\frac{r_K^2(\theta)}{r_L(\theta)}\cdot\big(r_L(\theta)-r_K(\theta)\big)\cdot m^{-2}(\theta)\mathrm{d}\theta$$

$$\lim_{n\to\infty}\left(\frac{n}{\ln n}\right)^2 \cdot \delta_H\left(K, R_n\right) = \frac{1}{8}\max_{\theta\in[0,2\pi]}\left[\frac{r_K(\theta)}{r_L(\theta)}\cdot\big(r_L(\theta)-r_K(\theta)\big)\right]\cdot m^{-2}(\theta).$$

Theorem 4. *Let L be centrally symmetric. Then with probability* 1,

$$\lim_{n\to\infty} n^2 \cdot \delta_P\left(K, R_{(n)}\right) = \frac{1}{4}\int_0^{2\pi}\frac{r_K^3(\theta)-3r_K^2(\theta)r_L(\theta)+2r_K(\theta)r_L^2(\theta)}{r_L^2(\theta)}\cdot m^{-2}(\theta)\mathrm{d}\theta$$

$$\lim_{n\to\infty} n^2 \cdot \delta_A\left(K, R_{(n)}\right) = \frac{1}{4}\int_0^{2\pi}\frac{r_K^2(\theta)}{r_L(\theta)}\cdot\big(r_L(\theta)-r_K(\theta)\big)\cdot m^{-2}(\theta)\mathrm{d}\theta$$

$$\lim_{n\to\infty}\left(\frac{n}{\ln n}\right)^2 \cdot \delta_H\left(K, R_{(n)}\right) = \frac{1}{8}\max_{\theta\in[0,2\pi]}\left[\frac{r_K(\theta)}{r_L(\theta)}\cdot\big(r_L(\theta)-r_K(\theta)\big)\right]\cdot m^{-2}(\theta).$$

Note that Hölder's inequality implies that

$$\left(\int_0^{2\pi} f(\theta)m^{-2}(\theta)\mathrm{d}\theta\right)^{\frac{1}{3}}\cdot\left(\int_0^{2\pi} m(\theta)\mathrm{d}\theta\right)^{\frac{2}{3}} \geq \int_0^{2\pi}\big(f(\theta)m^{-2}(\theta)\big)^{\frac{1}{3}}\cdot m^{\frac{2}{3}}(\theta)\mathrm{d}\theta,$$

hence

$$\frac{1}{4}\int_0^{2\pi}\frac{r_K(\theta)}{r_L^2(\theta)}\cdot\big(r_L^2(\theta)-r_K^2(\theta)\big)\cdot m^{-2}(\theta)\mathrm{d}\theta \geq \frac{1}{4}\left(\int_0^{2\pi}\left[\frac{r_K(\theta)}{r_L^2(\theta)}\cdot\big(r_L^2(\theta)-r_K^2(\theta)\big)\right]^{\frac{1}{3}}\mathrm{d}\theta\right)^3,$$

where the equality stands exactly when m is proportional to the expression $r_K(\theta)/r_L^2(\theta)\cdot\big(r_L^2(\theta)-r_K^2(\theta)\big)$. This means that by choosing m as above, we see that the approximation obtained from the random construction differs from the best approximation only by a factor of 6. This is similarly true for the perimeter and area deviation in both the inscribed and circumscribed cases.

In the following sections, we define and determine the geometric limits necessary for the proof of the main theorems. In the inscribed case, we compute the limits directly, then define a notion of duality, which helps us express the geometric limits of the circumscribed case with the inscribed ones. We note that this means we avoid the need to directly calculate these limits, which seems not feasible. Afterwards, we use the tools introduced in [12] and [20] to prove the four theorems above. Lastly, we show that these results yield the original results dealing with linear convexity.

3. Inscribed cases

Let $\theta_0 \in [0, 2\pi]$, and for sufficiently small $\Delta\theta$, let $L + p$ be the translate of L containing $x_K(\theta_0)$ and $x_K(\theta_0 + \Delta\theta)$ in its boundary so that the shorter arc of $L+p$ between the points is in K (see Figure 1). Let $\Delta_l(\theta_0, \theta_0 + \Delta\theta)$ denote the difference of the arc length of ∂K and $\partial(L + p)$ between $x_K(\theta_0)$ and $x_K(\theta_0 + \Delta\theta)$, and $\Delta_A(\theta_0, \theta_0 + \Delta\theta)$ the area enclosed by the abovementioned curves, and $\Delta_H(\theta_0, \theta_0 + \Delta\theta)$ their Hausdorff distance.

Lemma 1. *With the notation introduced above, the following limits hold:*

$$\lim_{\Delta\theta \to 0} \frac{\Delta_l(\theta_0, \theta_0 + \Delta\theta)}{(\Delta\theta)^3} = \frac{1}{24} \cdot \frac{r_K(\theta_0)}{r_L^2(\theta_0)} \cdot \left(r_L^2(\theta_0) - r_K^2(\theta_0)\right)$$
$$\lim_{\Delta\theta \to 0} \frac{\Delta_A(\theta_0, \theta_0 + \Delta\theta)}{(\Delta\theta)^3} = \frac{1}{12} \cdot \frac{r_K^2(\theta_0)}{r_L(\theta_0)} \cdot \left(r_L(\theta_0) - r_K(\theta_0)\right).$$

Proof. Let $\theta = \theta_0 + \Delta\theta$, $x_0 = x_K(\theta_0)$ and $x = x_K(\theta)$. Let $l_K(x_0, x)$ and $l_L(x_0, x)$ denote the shorter arc lengths of ∂K and $\partial(L + p)$ between x_0 and x, respectively, and let $d = d(x_0, x)$ be the length of the line segment $\overline{x_0 x}$. Furthermore, let $A_K(x_0, x)$ and $A_L(x_0, x)$ denote the area of the smaller cap cut off of K and $L + p$, respectively, by the line $x_0 x$. Finally, let η and η' be the angles of the unit normal vectors to $\partial(L + p)$ at x_0 and x, respectively.

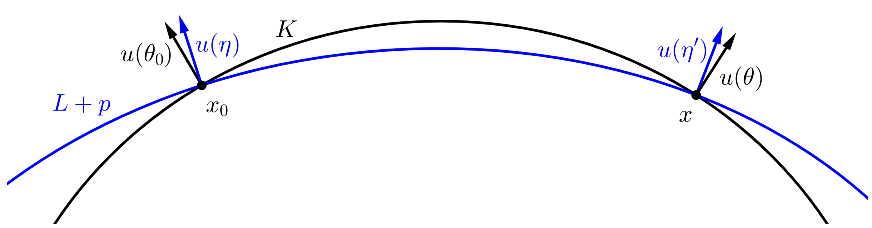

Figure 1.

Fix an arbitrary $\varepsilon > 0$. There exists some $\delta > 0$ that satisfies the following conditions:

(i) $\kappa_L(\theta_0)(1 + \varepsilon)^{-1} \le \kappa_L(\eta) \le \kappa_L(\theta_0)(1 + \varepsilon)$ whenever $|\theta_0 - \eta| < \delta$,

(ii) $(1+\varepsilon)^{-1} l_L(x_0, x) \le l_K(x_0, x) \le l_L(x_0, x)(1+\varepsilon)$ whenever $l_K(x_0, x) < \delta$, and

(iii)

$$a)\quad \frac{\kappa_L^2(\eta)}{24}\cdot(1+\varepsilon)^{-1}\leq\frac{l_L(x_0,x)-d}{l_L^3(x_0,x)}\leq\frac{\kappa_L^2(\eta)}{24}\cdot(1+\varepsilon)$$

$$b)\quad \frac{\kappa_L(\eta)}{12}\cdot(1+\varepsilon)^{-1}\leq\frac{A_L(x_0,x)}{l_L^3(x_0,x)}\leq\frac{\kappa_L(\eta)}{12}\cdot(1+\varepsilon)$$

whenever $l_L(x_0,x)<\delta$.

The existence of a δ satisfying (i) follows from the continuity of the curvature, (ii) from the limit $l_K(x_0,x)/l_L(x_0,x)\to 1$ as $l_K(x_0,x)\to 0$ (see [9]). Finally, (iii) arises from the limits established in Lemmas 1 and 2 of [9] (see also [12]), which are uniform in η due to the compactness of the domain.

Note that the unit normal vectors $u(\eta)$ and $u(\eta')$ fall on the shorter arc between $u(\theta_0)$ and $u(\theta)$ (see Figure 1), and clearly $l_L(x_0,x)\leq l_K(x_0,x)$. Thus choosing the point $x\in\partial K$ for which $l_K(x_0,x)<\delta$ and $|\theta_0-\theta|<\delta$ hold yields that all three requisites in (i)-(iii) are met. Now, applying (ii) to the expression $(l_L(x_0,x)-d)/l_K^3(x_0,x)$ yields

$$(1+\varepsilon)^{-3}\cdot\frac{l_L(x_0,x)-d}{l_L^3(x_0,x)}\leq\frac{l_L(x_0,x)-d}{l_K^3(x_0,x)}\leq\frac{l_L(x_0,x)-d}{l_L^3(x_0,x)}\cdot(1+\varepsilon)^3.$$

Using (iii)(a), we have

$$(1+\varepsilon)^{-4}\cdot\frac{\kappa_L^2(\eta)}{24}\leq\frac{l_L(x_0,x)-d}{l_K^3(x_0,x)}\leq\frac{\kappa_L^2(\eta)}{24}\cdot(1+\varepsilon)^4.$$

and lastly, by (i),

$$(1+\varepsilon)^{-6}\cdot\frac{\kappa_L^2(\theta_0)}{24}\leq\frac{l_L(x_0,x)-d}{l_K^3(x_0,x)}\leq\frac{\kappa_L^2(\theta_0)}{24}\cdot(1+\varepsilon)^6.$$

Now, it is clear that $l_K(x_0,x)/\Delta\theta\to r_K(\theta_0)$ as $\Delta\theta\to 0$, hence the above inequalities yield that

$$\lim_{\Delta\theta\to 0}\frac{l_L(x_0,x)-d}{(\Delta\theta)^3}=\frac{\kappa_L^2(\theta_0)}{24}\cdot r_K^3(\theta_0)=\frac{1}{24}\cdot\frac{r_K^3(\theta_0)}{r_L^2(\theta_0)},$$

and from the abovementioned lemma from [9] we have

$$\lim_{\Delta\to 0}\frac{l_K(x_0,x)-d}{(\Delta\theta)^3}=\frac{\kappa_K^2(\theta_0)}{24}\cdot r_K^3(\theta_0)=\frac{r_K(\theta_0)}{24}.$$

This yields the first part of the lemma after rewriting the expression as $\Delta_l(\theta_0,\theta_0+\Delta\theta)=(l_K(x_0,x)-d)-(l_L(x_0,x)-d)$.

The second part of the lemma follows similarly: using (ii), (iii)(b) and (i) in succession,

$$(1+\varepsilon)^{-5}\cdot\frac{\kappa_L(\theta_0)}{12}\leq\frac{A_L(x_0,x)}{l_K^3(x_0,x)}\leq\frac{\kappa_L(\theta_0)}{12}\cdot(1+\varepsilon)^5,$$

and we obtain the limit by $\Delta_A(\theta_0,\theta_0+\Delta\theta)=A_K(x_0,x)-A_L(x_0,x)$.

□

Lemma 2. *With the notation introduced at the beginning of the chapter, the following limit holds:*

$$\lim_{\Delta\theta\to 0}\frac{\Delta_H(\theta_0,\theta_0+\Delta\theta)}{(\Delta\theta)^2}=\frac{1}{8}\cdot\frac{r_K(\theta_0)}{r_L(\theta_0)}\cdot\big(r_L(\theta_0)-r_K(\theta_0)\big)$$

Proof. Without loss of generality, we may assume that $\theta_0 = 3\pi/2$ and $x(\theta_0)$ is the origin, thus the x-axis of the coordinate system is tangent to K at the origin, and K lies in the upper half plane. Let C_K be the osculating circle of ∂K at the origin. Let x be an arbitrary point of ∂K in the first quadrant, and $\hat{x}$ the point of C_K with the same abscissa as x, see Figure 2. Consider the circle C_L with radius $r_L(\theta_0)$ containing x_0 and x, and $\widehat{C_L}$ a copy of C_L rotated around x_0 containing $\hat{x}$. We denote the shorter arc of C_K between x_0 and $\hat{x}$ by c_K, the shorter arc of C_L between x_0 and $\hat{x}$ by c_L, and the shorter arc of $\widehat{C_L}$ between x_0 and x by $\widehat{c_L}$, as shown in the figure below. Lastly, we denote the arc of ∂K between x_0 and x by s_K, and the arc of $\partial(L+p)$ by s_L. By the triangle inequality of the Hausdorff metric, we have that

$$\delta_H(c_K, \widehat{c_L}) \leq \delta_H(c_K, s_K) + \delta_H(s_K, s_L) + \delta_H(s_L, c_L) + \delta_H(c_L, \widehat{c_L}),$$

which, along with a similar argument on $\delta_H(s_K, s_L)$, yields

$$\begin{aligned} - \delta_H(s_K, c_K) + \delta_H(c_K, \widehat{c_L}) - \delta_H(\widehat{c_L}, c_L) - \delta_H(c_L, s_L) \leq \\ \leq \delta_H(s_K, s_L) \leq \\ \leq \delta_H(s_K, c_K) + \delta_H(c_K, \widehat{c_L}) + \delta_H(\widehat{c_L}, c_L) + \delta_H(c_L, s_L). \end{aligned} \quad (2)$$

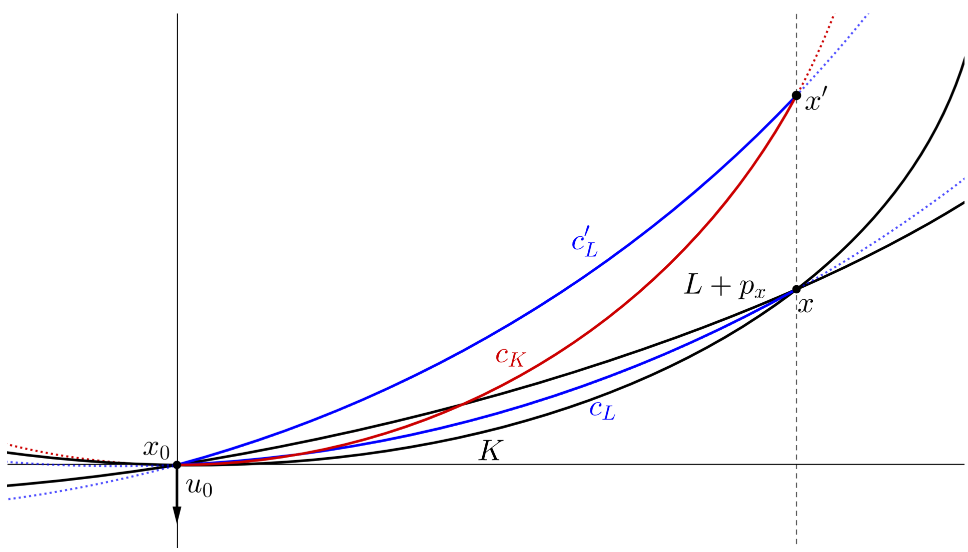

FIGURE 2.

The limit below follows directly from the relating result in [9]:

$$\lim_{x \to x_0} \frac{\delta_H(K[x_0, x], c)}{l_K^2(x_0, x)} = \frac{1}{8} \cdot \left| \frac{1}{R} - \kappa_K(\nu_0) \right|, \quad (3)$$

where $K[x_0, x]$ is the shorter arc between x and $x_0 = x_K(\nu_0)$, and c is the shorter circular arc of radius R connecting the same two points. This immediately yields that

$$\lim_{\Delta\theta \to 0} \frac{\delta_H(\widehat{c_L}, c_K)}{(\Delta\theta)^2} = \frac{1}{8} \left(\kappa_K(\theta_0) - \frac{1}{r_L(\theta_0)} \right) \cdot r_K^2(\theta_0) = \frac{1}{8} \cdot \frac{r_K(\theta_0)}{r_L(\theta_0)} \big(r_L(\theta_0) - r_K(\theta_0) \big).$$

Now, we will show that the remaining three Hausdorff distances used in the estimates in (2) tend to 0 when divided by x^2 as $x \to 0$, and equivalently when divided by $(\Delta\theta)^2$, as $\Delta\theta \to 0$. The latter equivalence comes from the observation that $\lim_{x\to 0} x/l(c_K) = 1$ (see [9]), hence $\lim_{x\to 0} \Delta\theta/x = \kappa_K(\theta_0) \neq 0$.

First, let x be sufficiently close to the origin so that s_K and c_K can be represented as graphs of twice differentiable functions f_1 and f_2. By the positioning of K and thus its osculating circle, we have that $f_1(0) = f_2(0) = f_1'(0) = f_2'(0) = 0$, thus by Lemma 3 (iii) of [9], we have that

$$\lim_{x\to 0^+} \frac{\delta_H(f_1[0,x],\ f_2[0,x])}{x^2} = \frac{f_1''(0) - f_2''(0)}{2},$$

where $f_i[0,x]$ denotes the graph of the function f_i on the interval $[0,x]$ $(i = 1, 2)$. As $f_1''(0) = f_2''(0) = \kappa_K(\theta_0)$, the limit above is 0. Furthermore, note that Taylor's theorem yields that $f_1(x) - f_2(x) = o(x^2)$.

Second, let c_L and $\widehat{c_L}$ be represented as the graphs of the functions g_1 and g_2. It is clear from the definition of the Hausdorff distance that $\delta_H(g_1[0,x], g_2[0,x]) \leq \max_{t\in[0,x]} |g_1(t)-g_2(t)|$, and it's easy to see from the contruction that the maximum is obtained at $t = x$. This yields that

$$\max_{t\in[0,x]} |g_1(t) - g_2(t)| = g_1(x) - g_2(x) = f_1(x) - f_2(x) = o(x^2),$$

and hence

$$0 \leq \lim_{x\to 0} \frac{\delta_H(g_1[0,x], g_2[0,x])}{x^2} \leq \lim_{x\to 0} \frac{\max_{t\in[0,x]} |g_1(t) - g_2(t)|}{x^2} = \lim_{x\to 0} \frac{o(x^2)}{x^2} = 0.$$

Lastly, we examine the limit $\delta(c_L, s_L)/l_L^2(x_0, x)$ as $l_L(x_0, x) \to 0$. Fix an arbitrary $\varepsilon > 0$. There exists some $\delta > 0$ that satisfies the following conditions:

(i) $|\kappa_L(\theta_0) - \kappa_L(\eta)| < \varepsilon$ whenever $|\theta_0 - \eta| < \delta$,

(ii) $(1+\varepsilon)^{-1} l_L(x_0, x) \leq l_K(x_0, x) \leq l_L(x_0, x)(1+\varepsilon)$ whenever $l_K(x_0, x) < \delta$, and

(iii)

$$(1+\varepsilon)^{-1} \cdot \frac{|\kappa_L(\theta_0) - \kappa_L(\eta)|}{8} \leq \frac{\delta_H(s_L, c_L)}{l_L^2(x_0, x)} \leq \frac{|\kappa_L(\theta_0) - \kappa_L(\eta)|}{8} \cdot (1+\varepsilon)$$

whenever $l_L(x_0, x) < \delta$.

The existence of such δ for (i) and (ii) are obtained similarly to the previous proof, and (iii) from the limit in (3), which is uniform in η. Similarly to the proof of Lemma 1, choosing the point $x \in \partial K$ so that $l_K(x_0, x) < \delta$ and $|\theta_0 - \theta| < \delta$ yields that all three requisites in (i)-(iii) are met.

Applying (ii) to the expression $\delta_H(s_L, c_L)/l_K^2(x_0, x)$ gives us

$$(1+\varepsilon)^{-2} \cdot \frac{\delta_H(s_L, c_L)}{l_L^2(x_0, x)} \leq \frac{\delta_H(s_L, c_L)}{l_K^2(x_0, x)} \leq \frac{\delta_H(s_L, c_L)}{l_L^2(x_0, x)} \cdot (1+\varepsilon)^2$$

Now, using (iii), we get

$$(1+\varepsilon)^{-3} \cdot \frac{|\kappa_L(\theta_0) - \kappa_L(\eta)|}{8} \leq \frac{\delta_H(s_L, c_L)}{l_K^2(x_0, x)} \leq \frac{|\kappa_L(\theta_0) - \kappa_L(\eta)|}{8} \cdot (1+\varepsilon)^3$$

Finally, applying (i) yields

$$(1+\varepsilon)^{-3} \cdot \frac{\varepsilon}{8} \leq \frac{\delta_H(s_L, c_L)}{l_K^2(x_0, x)} \leq \frac{\varepsilon}{8} \cdot (1+\varepsilon)^3,$$

hence

$$\lim_{\Delta\theta\to 0}\frac{\delta_H(s_L,c_L)}{l_K^2(x_0,x)}=0.$$

The four limits determined above yield the assertion using (2). □

4. An L-convex Duality

Let L be a centrally symmetric convex disc centered at the origin, i.e. $L=-L$, and let K be an L-convex disc. We define the *L-convex dual* of K as

$$K^*=\{y\mid K\subseteq y+L\}.$$

Using the symmetry of L, this can be reformulated as

$$K^*=\bigcap_{x\in K}L+x. \tag{4}$$

This also yields that K^* is the intersection of translates of L, hence is L-convex as well. This is a generalization of a duality defined in [8] for spindle-convex discs.

To describe the relationship between K and K^*, we need to define the following notion. The area of the Minkowski sum $\lambda_1K_1+\lambda_2K_2$ for any $\lambda_1,\lambda_2>0$ can be written as

$$A(\lambda_1K_1+\lambda_2K_2)=\lambda_1^2A(K_1)+2\lambda_1\lambda_2A(K_1,K_2)+\lambda_2^2A(K_2),$$

where $A(K_1,K_2)$ is called the *mixed area* of K_1 and K_2. The notion of mixed volumes can be similarly defined in higher dimensions, see [18].

Lemma 3. *For any L-convex disc K, the following hold:*

- (i) $K+(-K^*)=L$
- (ii) $h_K(u)+h_{K^*}(-u)=h_L(u)$
- (iii) $r_K(u)+r_{K^*}(-u)=r_L(u)$
- (iv) $\mathrm{Per}(K)+\mathrm{Per}(K^*)=\mathrm{Per}(L)$
- (v) $A(K^*)=A(L)-2A(K,L)+A(K)$

Proof. Fix a direction $u\in\mathcal{S}^1$, $x=x_K(u)$, and let x^*+L be the supporting L of K at $x\in\partial K$. Then by the definition of K^*, we have $x^*\in K^*$, and by (4), $x^*\in x+L$. It follows that x^* is a boundary point of K^*, specifically $x^*=x_{K^*}(-u)$, and it's support L is $x+L$. This yields (ii) and consequently (i). Furthermore, as $h_K(u)+h''_K(u)=r_K(u)$, (iii) immediately follows from (ii).

The proof of (iv) and (v) rest on the following well-known expressions (see Chapter 1 of [17]):

$$\mathrm{Per}(K)=\int_0^{2\pi}h_K(\theta)\mathrm{d}\theta \tag{5}$$

$$A(K)=\frac{1}{2}\int_0^{2\pi}h_K(\theta)\cdot\big(h_K(\theta)+h''_K(\theta)\big)\mathrm{d}\theta=\frac{1}{2}\int_0^{2\pi}h_K(\theta)r_K(\theta)\mathrm{d}\theta \tag{6}$$

$$A(K_1,K_2)=\frac{1}{2}\int_0^{2\pi}h_{K_1}(\theta)\cdot\big(h_{K_2}(\theta)+h''_{K_2}(\theta)\big)=\frac{1}{2}\int_0^{2\pi}h_{K_1}(\theta)r_{K_2}(\theta)\mathrm{d}\theta \tag{7}$$

From (5) and (ii), (iv) clearly follows. Now, note that in (7), K_1 and K_2 may be interchanged, hence

$$A(K_1,K_2)=\frac{1}{4}\left(\int_0^{2\pi}h_{K_1}(\theta)r_{K_2}(\theta)\mathrm{d}\theta+\int_0^{2\pi}h_{K_2}(\theta)r_{K_1}(\theta)\mathrm{d}\theta\right).$$

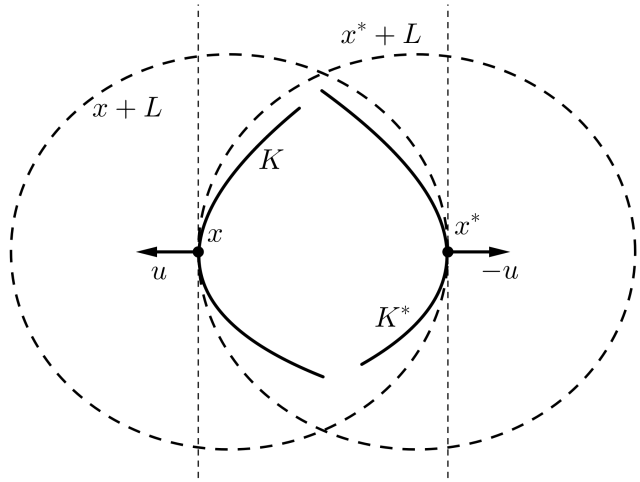

FIGURE 3.

This yields that

$$A(K^*) = \frac{1}{2}\int_0^{2\pi} h_{K^*}(\theta) r_{K^*}(\theta) \mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi} (h_L(\theta) - h_K(\theta))(r_L(\theta) - r_K(\theta)) \mathrm{d}\theta =$$

$$= \frac{1}{2}\int_0^{2\pi} h_L(\theta) r_L(\theta) - \frac{1}{2}\int_0^{2\pi} h_L(\theta) r_K(\theta) + h_K(\theta) r_L(\theta) \mathrm{d}\theta + \frac{1}{2}\int_0^{2\pi} h_K(\theta) r_K(\theta) \mathrm{d}\theta =$$

$$= A(L) - 2A(K, L) + A(K).$$

□

5. CIRCUMSCRIBED CASES

Let L be a centrally symmetric convex disc centered at the origin, and K an L-convex disc. Let $\theta_0 \in [0, 2\pi]$, and for sufficiently small $\Delta\theta$, let $L + x_0^*$ and $L + x^*$ be the translates of L supporting K at $x_0 = x_K(\theta_0)$ and $x = x_K(\theta_0 + \Delta\theta)$, respectively. Furthermore, we denote the intersection of $\partial(L + x_0^*)$ and $\partial(L + x^*)$ on the shorter arc between $u_0 = u(\theta_0)$ and $u = u(\theta_0 + \Delta\theta)$ by y, see Figure 4. We use the notation Γ for the union of $\partial(L + x_0^*)$ between x_0 and y, and $\partial(L + x^*)$ between x and y. Lastly, let $\Delta_l(\theta_0, \theta_0 + \Delta\theta)$, $\Delta_A(\theta_0, \theta_0 + \Delta\theta)$ and $\Delta_H(\theta_0, \theta_0 + \Delta\theta)$ be the difference of the arc length of Γ and ∂K between x_0 and x, the area enclosed by the two curves, and their Hausdorff distance, respectively.

Consider the disc $\widehat{K}$ obtained by taking the union of K and the cap determined by the curve Γ. It is easy to see that $\widehat{K}$ is L-convex as well. Moreover, it is also clear that Δ_l, Δ_A and Δ_H can be expressed, respectively, as the difference of perimeter, area, and the Hausdorff distance of $\widehat{K}$ and K.

Consider the L-convex dual of $\widehat{K}$ and K. It is clear from the definition that $K \subseteq \widehat{K}$ implies $\widehat{K}^* \subseteq K^*$. For every $v \in \mathcal{S}^1$ on the longer arc of $\mathcal{S}^1$ determined by $u(\theta_0)$ and $u(\theta)$, we have $x_K(v) = x_{\widehat{K}}(v)$, and hence $x_{K^*}(-v) = x_{\widehat{K}^*}(-v)$. Finally, $y + L$ contains both x^* and x_0^* in its boundary, thus $\widehat{K}^*$ and K^* describe the geometric properties inspected in the inscribed case.As mentioned in the introduction, this means that by using duality and its properties, we can rely on the limits obtained in the inscribed case when examining the circumscribed ones.

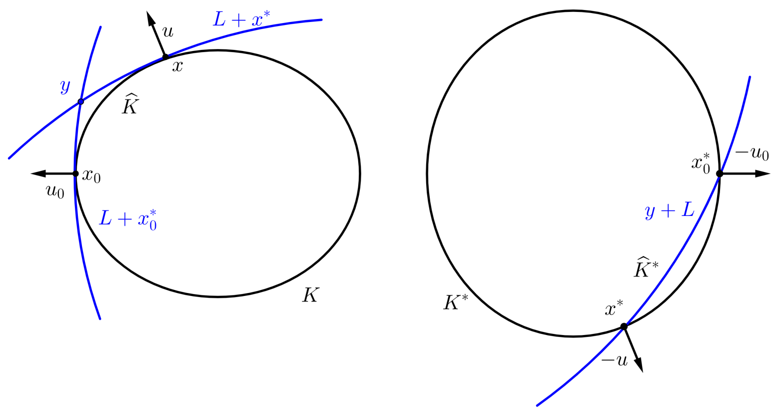

FIGURE 4.

Lemma 4. *Let L be centrally symmetric, centered at the origin. With the notation above,*

$$\lim_{\Delta\theta\to 0}\frac{\Delta_l(\theta_0,\theta_0+\Delta\theta)}{(\Delta\theta)^3}=\frac{1}{24}\cdot\frac{r_K^3(\theta_0)-3r_K^2(\theta_0)r_L(\theta_0)+2r_K(\theta_0)r_L^2(\theta_0)}{r_L^2(\theta_0)}$$
$$\lim_{\Delta\theta\to 0}\frac{\Delta_A(\theta_0,\theta_0+\Delta\theta)}{(\Delta\theta)^3}=\frac{1}{24}\cdot\frac{r_K^2(\theta_0)}{r_L(\theta_0)}\cdot\big(r_L(\theta_0)-r_K(\theta_0)\big)$$
$$\lim_{\Delta\theta\to 0}\frac{\Delta_H(\theta_0,\theta_0+\Delta\theta)}{(\Delta\theta)^2}=\frac{1}{8}\cdot\frac{r_K(\theta_0)}{r_L(\theta_0)}\cdot(r_L(\theta_0)-r_K(\theta_0)).$$

Proof. The limits for length and Hausdorff distance immediately follow from the formulae relating K and K^*, and the limits established in the inscribed cases. For the arc length, we get

$$\begin{aligned}\lim_{\Delta\theta\to 0}\frac{\mathrm{Per}(\widehat{K})-\mathrm{Per}(K)}{(\Delta\theta)^3}&\overset{\text{(Lemma 3)}}{=}\lim_{\Delta\theta\to 0}\frac{\mathrm{Per}(K^*)-\mathrm{Per}(\widehat{K}^*)}{(\Delta\theta)^3}=\\&\overset{\text{(Lemma 1.)}}{=}\frac{1}{24}\cdot\frac{r_{K^*}(\pi+\theta_0)}{r_L^2(\pi+\theta_0)}\cdot\big(r_L^2(\pi+\theta_0)-r_{K^*}^2(\pi+\theta_0)\big)=\\&\overset{\text{(Lemma 3)}}{=}\frac{1}{24}\cdot\frac{r_L(\theta_0)-r_K(\theta_0)}{r_L^2(\theta_0)}\cdot r_K(\theta_0)\cdot\big(2r_L(\theta_0)-r_K(\theta_0)\big)=\\&=\frac{r_K^3(\theta_0)-3r_K^2(\theta_0)r_L(\theta_0)+2r_K(\theta_0)r_L^2(\theta_0)}{24r_L^2(\theta_0)}.\end{aligned}$$

It readily follows from Lemma 3 (ii) that $h_{\widehat{K}}(u)-h_K(u)=h_{K^*}(-u)-h_{\widehat{K}^*}(-u)$, hence by the relationship between the Hausdorff distance of sets and their support functions (see (1)), we have $\delta_H(K,\widehat{K})=\delta_H(K^*,\widehat{K}^*)$. This observation, along with the inscribed limit in Lemma 2, gives us

$$\lim_{\Delta\theta\to 0}\frac{\delta_H(K,\widehat{K})}{(\Delta\theta)^2}=\lim_{\Delta\theta\to 0}\frac{\delta_H(K^*,\widehat{K}^*)}{(\Delta\theta)^2}= \\ =\frac{1}{8}\cdot\frac{r_{K^*}(\pi+\theta_0)}{r_L(\pi+\theta_0)}\cdot\big(r_L(\pi+\theta_0)-r_{K^*}(\pi+\theta_0)\big)= \\ =\frac{1}{8}\cdot\frac{r_L(\theta_0)-r_K(\theta_0)}{r_L(\theta_0)}\cdot r_K(\theta_0).$$

Now we turn to the limit concerning the area. By Lemma 3 (v) and (7), we have

$$A(\widehat{K})-A(K)=A(\widehat{K}^*)-A(K^*)-2\big(A(\widehat{K}^*,L)-A(K^*,L)\big)= \\ =-\left[\Big(A(K^*)-A(\widehat{K}^*)\Big)-\int_0^{2\pi}\big(h_{K^*}(\nu)-h_{\widehat{K}^*}(\nu)\big)r_L(\nu)\mathrm{d}\nu\right]. \quad (8)$$

Note that from Lemma 1,

$$\lim_{\Delta\theta\to 0}\frac{A(\widehat{K}^*)-A(K^*)}{(\Delta\theta)^3}=\frac{1}{12}\cdot\frac{r^2_{K^*}(\pi+\theta_0)}{r_L(\pi+\theta_0)}\cdot\big(r_L(\pi+\theta_0)-r_{K^*}(\pi+\theta_0)\big)$$

clearly follows.

With a slight abuse of words, we say that the angle β is *between* α_1 and α_2 exactly when $u(\beta)$ is on the shorter arc of $\mathcal{S}^1$ between $u(\alpha_1)$ and $u(\alpha_2)$. For the sake of compactness, we also introduce the notation $\nu_0=\pi+\theta_0$ and $\nu=\pi+\theta$, and we clearly have $\Delta\nu=\Delta\theta$.

Now, observe that for any ψ not between ν_0 and ν, we have $h_{\widehat{K}^*}(\psi)-h_{K^*}(\psi)=0$, thus the integral in (8) may be considered only on the interval between them, which we denote by i. Furthermore, for any $\varepsilon>0$, there exists some δ, where for every ν satisfying $|\nu_0-\nu|<\delta$, the inequality $(1+\varepsilon)^{-1}r_L(\nu_0)\le r_L(\psi)\le r_L(\nu_0)(1+\varepsilon)$ holds for every ψ between ν_0 and ν. This implies that

$$(1+\varepsilon)^{-1}r_L(\nu_0)\int_i\big(h_{K^*}(\psi)-h_{\widehat{K}^*}(\psi)\big)\mathrm{d}\psi\le \\ \le\int_i\big(h_{K^*}(\psi)-h_{\widehat{K}^*}(\psi)\big)r_L(\psi)\mathrm{d}\psi\le \\ \le(1+\varepsilon)r_L(\nu_0)\int_i\big(h_{K^*}(\psi)-h_{\widehat{K}^*}(\psi)\big)\mathrm{d}\psi \quad (9)$$

holds whenever ν is sufficiently close to ν_0.

Now, the integration formula for the perimeter (5) along with Lemma 1 yields that

$$\lim_{\Delta\theta\to 0}\frac{\int_0^{2\pi}\big(h_{\widehat{K}^*}(\psi)-h_{K^*}(\psi)\big)\mathrm{d}\psi}{(\Delta\theta)^3}=\frac{1}{24}\cdot\frac{r_{K^*}(\nu_0)}{r^2_L(\nu_0)}\cdot\big(r^2_L(\nu_0)-r^2_{K^*}(\nu_0)\big),$$

and hence by (9),

$$\lim_{\Delta\theta\to 0}\frac{\int_0^{2\pi}\big(h_{\widehat{K}^*}(\psi)-h_{K^*}(\psi)\big)r_L(\psi)\mathrm{d}\psi}{(\Delta\theta)^3}=\frac{1}{24}\cdot\frac{r_{K^*}(\nu_0)}{r_L(\nu_0)}\cdot\big(r^2_L(\nu_0)-r^2_{K^*}(\nu_0)\big).$$

Note that $r_L(\nu)=r_L(\theta)$ due to the symmetry of L, and with our current notation, $r_K(\theta_0)=r_L(\nu_0)-r_{K^*}(\nu_0)$ by Lemma 3. Using the previous observations, the limit established in the claim of the lemma is obtained as follows:

$$\lim_{\Delta\theta\to 0}\frac{A(\widehat{K})-A(K)}{(\Delta\theta)^3}=-\lim_{\Delta\theta\to 0}\left[\frac{A(K^*)-A(\widehat{K}^*)}{(\Delta\theta)^3}-\frac{\int_0^{2\pi}\left(h_{K^*}(\psi)-h_{\widehat{K}^*}(\psi)\right)r_L(\psi)\mathrm{d}\psi}{(\Delta\theta)^3}\right]=$$
$$=-\frac{r_{K^*}^2(\nu_0)}{12r_L(\nu_0)}\left(r_L(\nu_0)-r_{K^*}(\nu_0)\right)+\frac{r_{K^*}(\nu_0)}{24r_L(\nu_0)}\left(r_L^2(\nu_0)-r_{K^*}^2(\nu_0)\right)=$$
$$=\frac{1}{24}\cdot\frac{r_{K^*}(\nu_0)}{r_L(\nu_0)}\left(r_L(\nu_0)-r_{K^*}(\nu_0)\right)^2=\frac{1}{24}\cdot\frac{r_L(\theta_0)-r_K(\theta_0)}{r_L(\theta_0)}\cdot r_K^2(\theta_0).$$

□

6. Proof of Theorems 1 and 2

To prove Theorems 1 and 2, we use the tools described in [12, Section 4] of McClure and Vitale, also summarized in [9, Section 3].

Let f be a real-valued function on an interval $[a,b]$, and let T_n denote a partition of $[a,b]$ of the form $T_n=(t_0,t_1,\dots,t_n)$ where $a=t_0<t_1<t_2<\ldots<t_n=b$. Consider a functional $E(f,T_n)$ that admits a decomposition of the following additive form, relative to T_n:

$$E(f,T_n)=\sum_{i=0}^{n-1}e(f,t_i,t_{i+1}).$$

Furthermore, let $E_n(f)=\inf_{T_n}E(f,T_n)$. The results will follow from the following three assumptions.

Assumption 1. For any (α,β) satisfying $a\le\alpha<\beta\le b$, $e(f,\alpha,\beta)\ge 0$. Further, $e(f,\cdot,\cdot)$ is sub-additive over contiguous subintervals of $[a,b]$, that is, if $a\le\alpha<\beta<\gamma\le b$, then

$$e(f,\alpha,\beta)+e(f,\beta,\gamma)\le e(f,\alpha,\gamma).$$

Assumption 2. There is a function J_f on $[a,b]$, associated with f, and a constant $m>1$ such that

(i) J_f is nonnegative and piecewise continuous on $[a,b]$, admitting at worst a finite number of jump discontinuities, and

(ii)
$$\lim_{h\to 0+}\frac{e(f,\alpha,\alpha+h)}{h^m}=J_f(\alpha+).$$
This limit is uniform in that the difference $|J_f(\alpha+)-e(f,\alpha,\alpha+h)/h^m|$ can be made uniformly small when $(\alpha,\alpha+h)$ is contained in an interval where J_f is continuous.

Assumption 3. $e(f,\alpha,\beta)$ depends continuously on (α,β).

Corollary 2.1 in [12] states the following.

Theorem 5 (McClure, Vitale [12]). *If Assumptions 1-3 hold for $e(f,\alpha,\beta)$, then*

$$\lim_{n\to\infty}n^{m-1}E_n(f)=\left(\int_a^b(J_f(s))^{1/m}\mathrm{d}s\right)^m.$$

Now, we use Theorem 5 to prove the formula for perimeter deviation in Theorem 1. Consider the partition $\Theta_n=(\theta_0,\theta_1,\dots,\theta_n)$ of the interval $[0,2\pi]$, and let P_n be the L-convex hull of the points $x_K(\theta_0),\dots,x_K(\theta_n)$. Setting $E(f,\Theta_n)=$

$\delta_P(K,P_n)$, and $e(f,\theta_i,\theta_{i+1}) = \Delta_l(\theta_i,\theta_{i+1})$ with the notation introduced in Section 3, we have

$$E(f,\Theta_n) = \sum_{i=0}^{n} e(f,\theta_i,\theta_{i+1}),$$

and $\delta_P(K,K_n^P) = E_n(f) = \inf_{\Theta_n} E(f,\Theta_n)$. Assumptions 1 and 3 clearly hold for $e(f,\cdot,\cdot)$. The function

$$J_f(\theta) = \frac{1}{24}\cdot\frac{r_K(\theta)}{r_L^2(\theta)}\cdot\left(r_L^2(\theta) - r_K^2(\theta)\right)$$

is continuous on $[0,2\pi]$, thus satisfying (i) of Assumption 2. The limit in (ii) holds by Lemma 1 with $m=3$, and is clearly uniform in $[0,2\pi]$ in the sense of Assumption 2. Hence by Theorem 5 and Lemma 1,

$$\lim_{n\to\infty} n^2 E_n(F) = \left(\int_0^{2\pi}(J_f(\theta))^{1/3}\mathrm{d}\theta\right)^3 = \frac{1}{24}\cdot\left(\int_0^{2\pi}\left[\frac{r_K(\theta)}{r_L^2(\theta)}\cdot\left(r_L^2(\theta)-r_K^2(\theta)\right)\right]^{1/3}\mathrm{d}\theta\right)^3$$

The formulae for area deviation in the inscribed case, and perimeter and area deviation in the circumscribed case are proven in the same way, using the limits acquired in Lemmas 1 and 4. For the formulae concerning the Hausdorff distance, we need a modified form of Theorem 5, also quoted from [12].

Consider a function $G(f,T_n)$ that admits a decomposition of the following form, relative to T_n:

$$G(f,T_n) = \max_{0\le i\le n-1} g(f,t_i,t_{i+1}).$$

Futhermore, let $G(T_n) = \inf_{T_n} G(f,T_n)$.

Assumption 4. For any (α,β) satisfying $a\le\alpha<\beta\le b$, $g(f,\alpha,\beta)\le 0$. Further, if $a\le\alpha<\beta<\gamma\le b$, then

$$\max(g(f,\alpha,\beta),\ g(f,\beta,\gamma)) \le g(f,\alpha,\gamma).$$

Lemma 5 in [12] states the following.

Theorem 6 (McClure, Vitale [12])**.** *If Assumptions 2-4 hold for $g(f,\alpha,\beta)$, then*

$$\lim_{n\to\infty} n^m G_n(f) = \left(\int_a^b (J_f(s))^{1/m}\mathrm{d}s\right)^m.$$

To prove the remainder of Theorem 1, we set $G(f,\Theta_n) = \delta_H(K,P_n)$ and $g(f,\theta_i,\theta_{i+1}) = \Delta_H(\theta_i,\theta_{i+1})$, hence Assumptions 3 and 4 clearly hold. By Lemma 2, we obtain the function $J_f(\theta)$ satisfying Assumption 3 with $m=2$. Thus Theorem 6 and Lemma 2 yield the formula for the Hausdorff distance in Theorem 1. Similarly, the Hausdorff distance formula in Theorem 2 is obtained with the help of Lemma 4.

7. Proof of Theorems 3 and 4

We follow the ideas of Schneider in [20].

Lemma 5 (Drobot [4], Schneider [20])**.** *Let $X_1,X_2,\dots,X_n$ be a sequence of independent random variables, uniformly distributed in $[0,1]$. For each n, let $X_1(n)\le X_2(n)\le\dots\le X_n(n)$ be the values of $X_1,X_2,\dots,X_n$ arranged in non-decreasing order. Define $U_j(n) = X_{j+1}(n) - X_j(n)$ $j=1,\dots,n$, where $X_{n+1}(n) = 1 + X_1(n)$.*

Let g be a continuous real valued function on $[0,1]$, and let $p>1$. Then with probability 1

$$\lim_{n\to\infty} n^{p-1}\sum_{j=1}^{n} g\left[X_j(n)\right]\left[U_j(n)\right]^p = \Gamma(p+1)\int_0^1 g(x)\mathrm{d}x.$$

Let $Y_1, Y_2, \dots, Y_n$ be a sequence of independent random variables on ∂K, distributed according to the density function μ, and let us define the function m by $m(\theta)=\mu(x_K(\theta))/\kappa_K(\theta)$. Let R_n denote the L-convex hull of $Y_1,\dots,Y_n$. We define a map $\Psi\colon[0,2\pi]\to[0,1]$ by

$$\Psi\colon \theta\mapsto \int_0^\theta m(t)\mathrm{d}t$$

and let $X_i=\Psi\circ x_K^{-1}\circ Y_i$. Then X_i's are independent uniform random points in $[0,1]$. We define $\theta_j(n)$ by $\Psi(\theta_j(n))=X_j(n)$ for every $j=1,\dots,n+1$. Then we clearly have

$$X_{j+1}(n)-X_j(n)=\int_{\theta_j(n)}^{\theta_{j+1}(n)} m(t)\mathrm{d}t.$$

By the uniform continuity of m, we obtain

$$X_{j+1}-X_j(n)=m(\theta_j(n))(\theta_{j+1}(n)-\theta_j(n))(1+o(1)) \tag{10}$$

with $o(1)\to 0$ for $|\theta_{j+1}(n)-\theta_j(n)|\to 0$, uniformly in $\theta_j(n)\in[0,2\pi]$.

With the notation introduced in Section 3, it is clear that

$$\delta_P(K,R_n)=\sum_1^n \Delta_P(\theta_j(n),\theta_{j+1}(n)).$$

This decomposition, along with Lemma 1 and (10), yields that $\delta_P(K,R_n)$ can be expressed as

$$\sum_1^n \frac{1}{24}\cdot\frac{r_K(\theta_j(n))}{r_L^2(\theta_j(n))}\cdot\left(r_L^2(\theta_j(n))-r_K^2(\theta_j(n))\right)\cdot\left(\theta_{j+1}(n)-\theta_j(n)\right)^3(1+o(1)) =$$

$$=\sum_1^n \frac{1}{24}\cdot\frac{r_K(\theta_j(n))}{r_L^2(\theta_j(n))}\cdot\left(r_L^2(\theta_j(n))-r_K^2(\theta_j(n))\right)\left(\frac{X_{j+1}(n)-X_j(n)}{m(\theta_j(n))}\right)^3(1+o(1)) \tag{11}$$

Now, $\theta_j(n)=\Psi^{-1}(X_j(n))$, so substituting

$$g(X)=\frac{r_K(\Psi^{-1}(X))}{24 r_L^2(\Psi^{-1}(X))}\cdot\frac{r_L^2(\Psi^{-1}(X))-r_K^2(\Psi^{-1}(X))}{m^3(\Psi^{-1}(X))}$$

into (11) we obtain that

$$\delta_P(K,R_n)=\sum_1^n g(X_j(n))U_j^3(n)(1+o(1)),$$

with $o(1)\to 0$ for $\max_j U_j(n)\to 0$. Hence Lemma 5 implies

$$\lim_{n\to\infty} n^2\cdot\delta_P(K,R_n)=\Gamma(4)\int_0^1 g(X)\mathrm{d}X.$$

By the definitions, $\Psi'=m$, hence substituting $X=\Psi(\theta)$ yields the first formula of Theorem 3. The formulae for area deviation in the inscribed case, and perimeter and area deviation in the circumscribed case are proven in the same way, using the limits acquired in Lemmas 1 and 4.

We now turn to proving the results regarding the Hausdorff distance.

Lemma 6. *Let $X_1, \dots, X_n$, $X_j(n)$, $U_j(n)$ and g be as in Lemma 5, and assume that $g(0) = g(1)$. Then*

$$\lim_{n\to\infty} \frac{n}{\ln n} \cdot \max_{1\le j\le n} g[X_j(n)]U_j(n) = \max_{x\in[0,1]} g(x).$$

By the definition of the Hausdorff distance, we clearly have that $\delta_H(K, R_n) = \max_j \Delta_H(\theta_j(n), \theta_{j+1}(n))$ with the notation introduced in Section 3. Using the limit obtained in Lemma 2, this can be expressed as

$$\max_j \frac{1}{8} \cdot \frac{r_K(\theta_j(n))}{r_L(\theta_j(n))} \cdot \big(r_L(\theta_j(n)) - r_K(\theta_j(n))\big) \cdot (\theta_{j+1}(n) - \theta_j(n))^2(1 + o(1)),$$

where $o(1) \to 0$ for $\max_j |\theta_{j+1}(n) - \theta_j(n)| \to 0$. This observation, along with (10) yields that $n/\ln n \cdot \delta_H^{1/2}(K, R_n)$ is written as

$$\frac{n}{\ln n} \cdot \max_j \left(\frac{r_K(\theta_j(n))}{8r_L(\theta_j(n))}\right)^{1/2} \big(r_L(\theta_j(n)) - r_K(\theta_j(n))\big)^{1/2} \cdot \frac{U_j(n)}{m(\theta_j(n))}(1 + o(1))$$

As $\theta_j(n) = \Psi^{-1}(X_j(n))$, setting

$$g(X) = \left(\frac{r_K(\Psi^{-1}(X))}{8r_L(\Psi^{-1}(X))}\right)^{1/2} \cdot \frac{\big(r_L(\Psi^{-1}(X)) - r_K(\Psi^{-1}(X))\big)^{1/2}}{m(\Psi^{-1}(X))}$$

and using Lemma 6 gets us to

$$\lim_{n\to\infty} \frac{n}{\ln n} \cdot \delta_H^{1/2}(K, R_n) = \lim_{n\to\infty} \frac{n}{\ln n} \max_j g(X_j(n))U_j(n)(1 + o(1)) = \max_{x\in[0,1]} g(x),$$

which yields the assertion for the Hausdorff distance in Theorem 3 after returning to the parameter θ and squaring the expressions. The corresponding result of Theorem 4 is obtained in the same way.

8. Connection to the previous results

In this section, we show that from the formula of 3 concerning the area yields the corresponding result in linear convexity by choosing $L = r\mathcal{B}^2$ and $r \to \infty$. The argument here closely follows the proof given in Section 3, [7].

Let $L = r\mathcal{B}^2$. Then L-convexity corresponds to spindle convexity, so in the proof we use the relevant terminology: we say r-disc-polygon instead of L-polygon, and r-convex hull instead of L-convex hull, which is here denoted by $[X]_r$.

Let R_n^r be a random r disc-polygon, and P_n the n-gon with the same vertices as R_n^r, and let

$$d^R(n) = (A(K) - A(R_n^r)) \cdot n^2 \quad ; \quad d(n) = (A(K) - A(P_n)) \cdot n^2;$$

$$I^R = \frac{1}{2}\int_0^{2\pi} \left(r_K^2(\theta) - \frac{r_K^3(\theta)}{r}\right) \cdot m^{-2}(\theta)\mathrm{d}\theta \quad ; \quad I = \frac{1}{2}\int_0^{2\pi} r_K^2(\theta) \cdot m^{-2}(\theta)\mathrm{d}\theta.$$

Fix $\varepsilon > 0$. Clearly, there exists some $R_1(\varepsilon)$ for which $1 - \varepsilon < I^r/I < 1$ holds for every $r > R_1(\varepsilon)$.

Elementary calculations show that there exists a $R_2(\varepsilon) \geq r_0$, depending only on K and ε, such that for all $r > R_2(\varepsilon)$ we have

$$\frac{A([p,q]_r)}{A([p,q]_{r_0}) - A([p,q]_r)} < \varepsilon, \tag{12}$$

for any point $p, q \in K$.

Let D^r_m be an r disc-polygon with m vertices, and P_m the m-gon with the same vertices. If $r > R_2(\varepsilon)$, then (12) yields

$$\begin{aligned} 1 < \frac{d(n)}{d^r(n)} &= 1 + \frac{A(R^r_n) - A(P_n)}{A(K) - A(R^r_n)} \\ &< 1 + \sup_{\substack{D^r_m \subseteq K, \\ 2 \leq m \leq n}} \frac{A(D^r_m) - A(P_m)}{A(D^{r_0}_m) - A(D^r_m)} < 1 + \varepsilon. \end{aligned}$$

In summary, for $R > \max(R_1(\varepsilon), R_2(\varepsilon))$ we have

$$(1-\varepsilon) \cdot \frac{d^R(n)}{I^R} < \frac{d(n)}{I} = \frac{d(n)}{d^R(n)} \cdot \frac{d^R(n)}{I^R} \cdot \frac{I^R}{I} < (1+\varepsilon) \cdot \frac{d^R(n)}{I^R},$$

which together with Theorem 3 yields that $d_n \to I$ almost surely as $n \to \infty$. The formulae of the perimeter and Hausdorff-distance follow in the same way.

9. Concluding remarks

A remaining open problem is to examine the circumscribed cases when L is not centrally symmetric. The proof used in the paper isn't applicable, and the direct computation of the necessary limits, as we've previously noted, seems intricate.

The same problem naturally arises in higher dimensions as well. The analytical and stochastic tools essentially don't generalise to higher dimensions, the necessary geometric notions are not yet fully developed, hence these are longer term questions.

Lastly, we note that the duality used in the paper, similarly to [8], holds interesting problems in itself, we only examined the tools necessary for our proof.

Declarations

Acknowledgements. K. Nagy was supported by the New National Excellence Program of the Ministry for Innovation and Technology from the source of the National Research, Development and Innovation Fund, grant no. ÚNKP-21-1-SZTE-331.

The authors were partially supported by Hungarian NKFIH grant FK135392.

This research was supported by the Ministry of Innovation and Technology of Hungary from the National Research, Development and Innovation Fund, project no. ITM NKFIA TKP2021-NVA-09.

Conflicts of interest/Competing interests. The authors have no relevant financial or non-financial interests to disclose.

Data availability. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] M. V. Balashov and E. S. Polovinkin, *M-strongly convex subsets and their generating sets*, Sbornik: Mathematics **191** (2000), no. 1, 25–60.

[2] K. Bezdek, Z. Lángi, M. Naszódi, and P. Papez, *Ball-polyhedra*, Discrete Comput. Geom. **38** (2007), no. 2, 201–230.

[3] E. M. Bronstein, *Approximation of convex sets by polytopes*, J Math Sci **153** (2008), 727–762.

[4] V. Drobot, *Probabilistic version of a curvature formula*, Ann. Prob **10** (1982), 620 – 622.

[5] L. Fejes Tóth, *Lagerungen in Der Ebene Auf Der Kugel Und Im Raum*, Vol. 65, Springer, 1953.

[6] F. Fodor, *Convex bodies and their approximations*, D.Sc. dissertation, 2020. `http://real-d.mtak.hu/1259/`.

[7] F. Fodor, P. Kevei, and V. Vígh, *On random disc polygons in smooth convex discs*, Adv. in Appl. Probab. **46** (2014), no. 4, 899–918.

[8] F. Fodor, Á. Kurusa, and V. Vígh, *Inequalities for hyperconvex sets*, Adv. in Geom. **16** (2016), no. 3, 337–348.

[9] F. Fodor and V. Vígh, *Disc-polygonal approximations of planar spindle convex sets*, Acta Sci. Math. (Szeged) **78** (2012), no. 1-2, 331–350.

[10] P. M. Gruber, *Comparisons of best and random approximation of convex bodies by polytopes*, Rend. Circ. Mat. Palermo (2) Suppl. **50** (1997), 189–216. II International Conference in "Stochastic Geometry, Convex Bodies and Empirical Measures" (Agrigento, 1996).

[11] Z. Lángi, M. Naszódi, and I. Talata, *Ball and spindle convexity with respect to a convex body*, Aequationes Math. **85** (2013), no. 1-2, 41–67.

[12] D. E. McClure and R. A. Vitale, *Polygonal Approximation of Plane Convex Bodies*, Journal of Mathematical Analysis and Applications **51** (1975), 326–358.

[13] M. Ludwig, *A Characterization of Affine Length and Asymptotic Approximation of Convex Discs*, Abh. Math. Sem. Univ. Hamburg **69** (1999), 75-88.

[14] E. S. Polovinkin, *Strongly Convex Analysis*, Sbornik: Mathematics **187** (1996), no. 2, 259–286.

[15] E. S. Polovinkin, *On Strongly Convex Sets and Strongly Convex Functions*, Journal of Mathematical Sciences **100** (2000), no. 6, 2633–2681.

[16] M. Reitzner, *Random Points on the Boundary of Smooth Convex Bodies*, Transactions of the American Mathematical Society **354** (2002), no. 6, 2243–2278.

[17] L. Santaló and M. Kac, *Integral Geometry and Geometric Probability*, Encyclopedia of Mathematics and its Applications, Cambridge: Cambridge University Press, 1976.

[18] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Second expanded edition, Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014.

[19] R. Schneider, *Discrete aspects of stochastic geometry*, Handbook of Discrete and Computational Geometry, 3rd ed., CRC Press, Boca Raton, 2017.

[20] R. Schneider, *Random approximation of convex sets*, Journal of Microscopy **151** (1988), 211 – 227.

Department of Geometry, Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, 6720 Szeged, Hungary

Email address: `kinga1204@live.com`

Department of Geometry, Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, 6720 Szeged, Hungary

Email address: `vigvik@math.u-szeged.hu`